Roshan Kakiya's Geometric Mean of the Cents of the Semitones with and without Inharmonicity for Balancing the Relationship between Harmonic and Inharmonic Coincident Upper Partials

Geometric Mean of the Cents of the Semitones without Inharmonicity for Balancing the Relationship between Harmonic Coincident Upper Partials

Two Coincident Partials are Upper Partials (Partial Number > 1)

Geometric Mean of the Semitones (Cents)
$= \langle\{[1200 \times \log_2(P_1)] / S_1\} \times \{[1200 \times \log_2(P_2)] / S_2\}\rangle^{1/N}$

P_1 is the partial number of the coincident partial of an interval's lower note.

S_1 is the number of semitones within P_1.

P_2 is the partial number of the coincident partial of an interval's upper note.

S_2 is the number of semitones within P_2.

N is the number of partials.

One Coincident Partial is an Upper Partial (Partial Number > 1)

Geometric Mean of the Semitones (Cents)
$= \{[1200 \times \log_2(P_1)] / S_1\}^{1/N}$

P_1 is the partial number of the coincident partial of an interval's lower note.

S_1 is the number of semitones within P_1.

N is the number of partials.

Example: Two Coincident Partials are Upper Partials (Partial Number > 1)

I have used the 3 : 2 Fifth A4-E5 for my calculations below.

P_1 = 3rd Partial of A4 = 3

S_1 = 19

P_2 = 2nd Partial of E5 = 2

S_2 = 12

N = 2

Geometric Mean of the Semitones (Cents)
$= \langle\{[1200 \times \log_2(3)] / 19\} \times \{[1200 \times \log_2(2)] / 12\}\rangle^{1/2}$ = 100.051434164 cents

3 : 2 Fifth A4-E5 (Cents)
$= 7 \times \langle\{[1200 \times \log_2(3)] / 19\} \times \{[1200 \times \log_2(2)] / 12\}\rangle^{1/2}$ = 700.360039147 cents

Harmonic 3rd Partial of A4 (Cents) = $1200 \times \log_2(3)$
= 1901.955000865 cents

Narrowed 3rd Partial of A4 (Cents) = $19 \times \langle\{[1200 \times \log_2(3)] / 19\} \times \{[1200 \times \log_2(2)] / 12\}\rangle^{1/2}$
= 1900.977249113 cents

Harmonic 3rd Partial of A4 (Cents) / Narrowed 3rd Partial of A4 (Cents)
= 1901.955000865 cents / 1900.977249113 cents = 1.000514342

Harmonic 2nd Partial of E5 (Cents) = $1200 \times \log_2(2)$
= 1200.000000000 cents

Widened 2nd Partial of E5 (Cents) = $12 \times \langle\{[1200 \times \log_2(3)] / 19\} \times \{[1200 \times \log_2(2)] / 12\}\rangle^{1/2}$
= 1200.617209966 cents

Widened 2nd Partial of E5 (Cents) / Harmonic 2nd Partial of E5 (Cents)
= 1200.617209966 cents / 1200.000000000 cents = 1.000514342

Example: One Coincident Partial is an Upper Partial (Partial Number > 1)

I have used the 2 : 1 Octave A4-A5 for my calculations below.

P_1 = 2nd Partial of A4 = 2

S_1 = 12

N = 1

Geometric Mean of the Semitones (Cents)
= $\{[1200 \times \log_2(2)] / 12\}^{1/1}$ = 100.000000000 cents

2 : 1 Octave A4-A5 (Cents)
= 12 × $\{[1200 \times \log_2(2)] / 12\}^{1/1}$ = 1200.000000000 cents

Geometric Mean of the Cents of the Semitones with Inharmonicity for Balancing the Relationship between Inharmonic Coincident Upper Partials

Two Coincident Partials are Upper Partials (Partial Number > 1)

Geometric Mean of the Semitones (Cents)
$= \langle\{[1200 \times \log_2(P_1 \times [\{1 + B_1 \times P_1^2\} / \{1 + B_1\}]^{1/2})] / S_1\} \times \{[1200 \times \log_2(P_2 \times [\{1 + B_2 \times P_2^2\} / \{1 + B_2\}]^{1/2})] / S_2\}\rangle^{1/N}$

P_1 is the partial number of the coincident partial of an interval's lower note.

B_1 is the inharmonicity coefficient of an interval's lower note.

S_1 is the number of semitones within P_1.

P_2 is the partial number of the coincident partial of an interval's upper note.

B_2 is the inharmonicity coefficient of an interval's upper note.

S_2 is the number of semitones within P_2.

N is the number of partials.

One Coincident Partial is an Upper Partial (Partial Number > 1)

Geometric Mean of the Semitones (Cents)
$= \{[1200 \times \log_2(P_1 \times [\{1 + B_1 \times P_1^2\} / \{1 + B_1\}]^{1/2})] / S_1\}^{1/N}$

P_1 is the partial number of the coincident partial of an interval's lower note.

B_1 is the inharmonicity coefficient of an interval's lower note.

S_1 is the number of semitones within P_1.

N is the number of partials.

Example: Two Coincident Partials are Upper Partials (Partial Number > 1)

I have used the 3 : 2 Fifth A4-E5 for my calculations below.

P_1 = 3rd Partial of A4 = 3

B_1 = Inharmonicity Coefficient of A4 = 0.000664

S_1 = 19

P_2 = 2nd Partial of E5 = 2

B_2 = Inharmonicity Coefficient of E5 = 0.001234

S_2 = 12

N = 2

Geometric Mean of the Semitones (Cents)
$= \langle\{[1200 \times \log_2(3 \times [\{1 + 0.000664 \times 3^2\} / \{1 + 0.000664\}]^{1/2})] / 19\} \times \{[1200 \times \log_2(2 \times [\{1 + 0.001234 \times 2^2\} / \{1 + 0.001234\}]^{1/2})] / 12\}\rangle^{1/2}$ = 100.305154788 cents

3 : 2 Fifth A4-E5 (Cents)
$= 7 \times \langle\{[1200 \times \log_2(3 \times [\{1 + 0.000664 \times 3^2\} / \{1 + 0.000664\}]^{1/2})] / 19\} \times \{[1200 \times \log_2(2 \times [\{1 + 0.001234 \times 2^2\} / \{1 + 0.001234\}]^{1/2})] / 12\}\rangle^{1/2}$ = 702.136083519 cents

Inharmonic 3rd Partial of A4 (Cents)
$= 1200 \times \log_2\{3 \times [(1 + 0.000664 \times 3^2) / (1 + 0.000664)]^{1/2}\}$ = 1906.537953837 cents

Narrowed 3rd Partial of A4 (Cents)
$= 19 \times \langle\{[1200 \times \log_2(3 \times [\{1 + 0.000664 \times 3^2\} / \{1 + 0.000664\}]^{1/2})] / 19\} \times \{[1200 \times \log_2(2 \times [\{1 + 0.001234 \times 2^2\} / \{1 + 0.001234\}]^{1/2})] / 12\}\rangle^{1/2}$ = 1905.797940979 cents

Inharmonic 3rd Partial of A4 (Cents) / Narrowed 3rd Partial of A4 (Cents)
= 1906.537953837 cents / 1905.797940979 cents = 1.000388296

Inharmonic 2nd Partial of E5 (Cents)
$= 1200 \times \log_2\{2 \times [(1 + 0.001234 \times 2^2) / (1 + 0.001234)]^{1/2}\}$ = 1203.194662329 cents

Widened 2nd Partial of E5 (Cents)
$= 12 \times \langle\{[1200 \times \log_2(3 \times [\{1 + 0.000664 \times 3^2\} / \{1 + 0.000664\}]^{1/2})] / 19\} \times \{[1200 \times \log_2(2 \times [\{1 + 0.001234 \times 2^2\} / \{1 + 0.001234\}]^{1/2})] / 12\}\rangle^{1/2}$ = 1203.661857461 cents

Widened 2nd Partial of E5 (Cents) / Inharmonic 2nd Partial of E5 (Cents)
= 1203.661857461 cents / 1203.194662329 cents = 1.000388296

Example: One Coincident Partial is an Upper Partial (Partial Number > 1)

I have used the 2 : 1 Octave A4-A5 for my calculations below.

P_1 = 2nd Partial of A4 = 2

B_1 = Inharmonicity Coefficient of A4 = 0.000664

S_1 = 12

N = 1

Geometric Mean of the Semitones (Cents)
$= \{[1200 \times \log_2(2 \times [\{1 + 0.000664 \times 2^2\} / \{1 + 0.000664\}]^{1/2})] / 12\}^{1/1}$
= 100.143454339 cents

2 : 1 Octave A4-A5 (Cents)
$= 12 \times \{[1200 \times \log_2(2 \times [\{1 + 0.000664 \times 2^2\} / \{1 + 0.000664\}]^{1/2})] / 12\}^{1/1}$
= 1201.721452071 cents

References

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