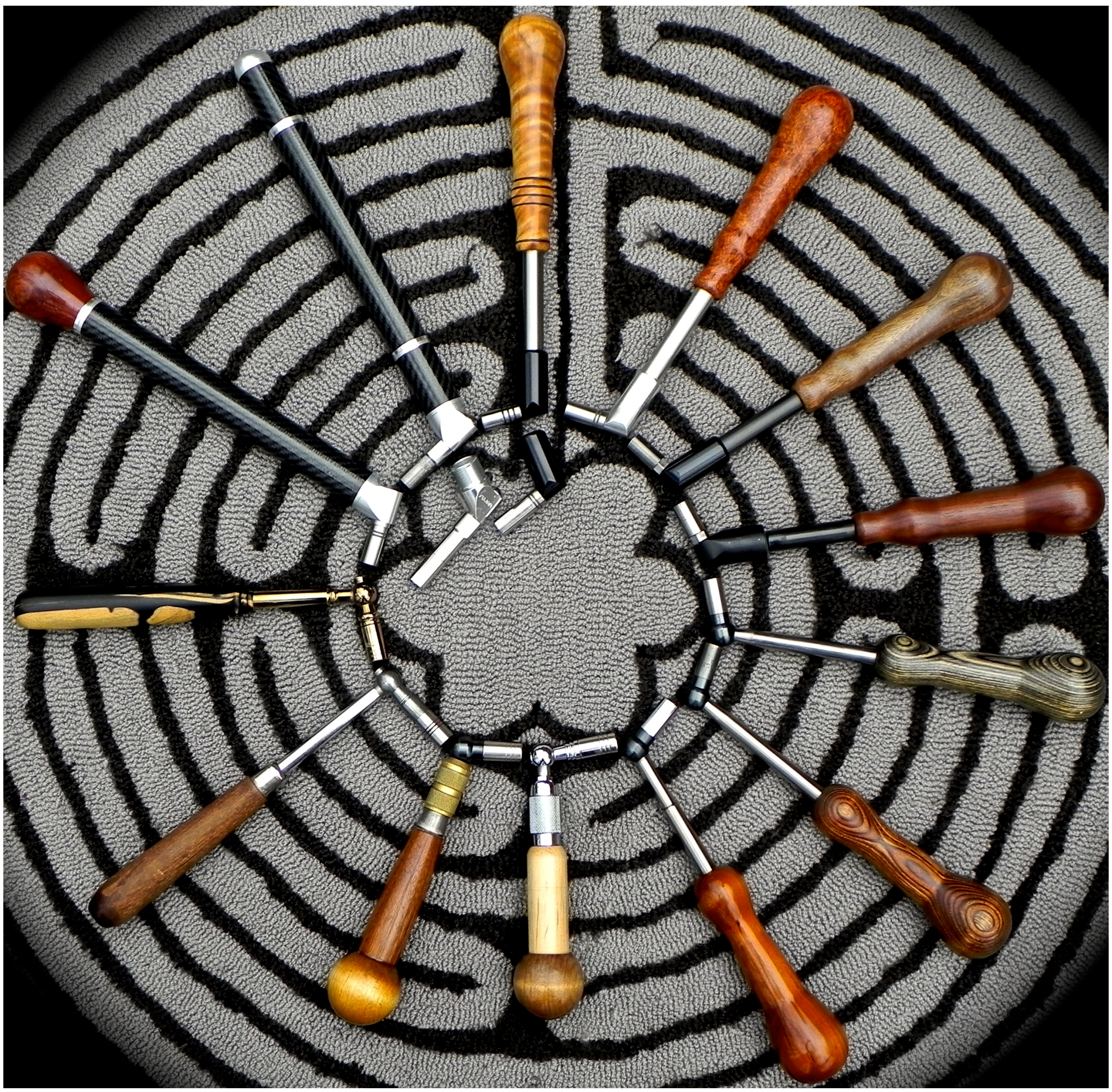
What If Pythagoras Got It Wrong?

I'm Kent Swafford and I am eager to find out what you all are going to teach me today, so let's get started! There is no handout, but the text from this class is available on the annual convention handout site. I may turn this into a series of Journal articles with online supplements, so stay tuned, if you are interested.

INTRODUCTION

It is said, "The cure for boredom is curiosity. There is no cure for curiosity."

I have tried many new things over the years as they have come along; some things have turned out great; some things have been disasters, but generally I don't talk about the disasters. I am no fanatic, but I may very well be a sucker for new things, like better tuning levers, for example.



Here are most of the devices I have owned on which to run CyberTuner.

And here is my current happy family of electronic tuning devices, no disasters here.

When you try something new, you have to meet it on its own terms, which was particularly difficult for some new tuning software I was asked to try out years ago now, because it required me to vastly improve my hammer technique in order take advantage of what it had to offer. So I worked for a number of months to bump up my hammer technique skills, and one day I finished a tuning with this new software, hit an arpeggio on the piano I was tuning — and I heard a sound that I had never heard before from a piano. And as I continued to tune more and more pianos with the new software and my new hammer technique, I continued to hear that sound. The sound can be described simply. It involves playing a keyboard length arpeggio with the damper pedal held down. After you hit the top note, you listen to the overall sound of the decay, and the sound can be pure, almost beat-free, dramatically so. So there is this effect that I know is real, and I have been on a quest ever since to explain this phenomenon; this quest has changed my life.

So, it turns out that there are tuners out there promoting various widths of equal temperament that they claim have a particularly pure overall sound, that is, when three or more notes are sounding, beats can be masked and the sound is pure. These various widths of equal temperament cannot all have the most pure sound, but they claim to, and I don't know and don't care which actually has the most pure overall sound. What I do know is that this pure sound derived simply from a wide equal temperament is real and is worth investigating and I have tried to do so, in open-minded fashion. Again I don't care which equal temperament has the best whole sound, so I am not promoting one over the other, but am talking about the concept of variable width equal temperament and its possible benefits which appear to be considerable.

I thought I should start with a reading list so you know a little about where I'm coming from, what I have put into this, and that I am not just making this stuff up out of whole cloth, other people did that first! Putting this class together gave me the perfect opportunity to check out some texts that I hadn't seen or hadn't seen recently. Books that I have reviewed for potential use in this class are:

Genesis of a Music, Harry Partch
Tuning and Temperament, J. Murray Barbour
Theory and Practice of Piano Tuning, Brian Capleton
The Craft of Piano Tuning, Daniel Levitan
Unequal Temperaments, 3rd Revised Edition, Claudio Di Veroli
On Pitch, Revised Edition, Rick Baldassin
New Techniques For Superior Aural Tuning, 2nd edition, Virgil Smith
A Clear and Practical Introduction to Temperament History, Fred Sturm
Historical Tuning of Keyboard Instruments, Robert Chuckrow
The New Tuning, Lucas Mason
Complete Course in Professional Piano Tuning, Floyd Stevens
(I had heard some quotes from this book and decided I should check it out. Let's see; I said I don't talk about disasters. Right?)
Five Lectures on the Acoustics of the Piano, Anders Askenfelt
Influences of Inharmonicity on Aural Tests in Equal Temperament, Gary Shulze
Stopper Tuning material from the web, Bernhard Stopper
CHAS, Circular Harmonic System, from the web, Alfredo Capurso
[Class handout] Dr. Al Sanderson

Ready to think? Ready to do this? We won't be, how you say, dumbing anything down here! I have a good bit of material to present. My aim here is to interest you in further study.

The History of Temperament and the Variability of Interval Widths

The tuning of musical instruments is a very big deal, so much so that some of the world's greatest scientists have turned their attention to the tuning of musical instruments. In fact, the subject of tuning musical instruments stretches back to the beginnings of both science and math. Speaking of the world's greatest scientists, here is a quote from one, namely, Stephen Hawking:

> According to legend, the first mathematical formulation of what we might today call a law of nature dates back to an Ionian named Pythagoras (ca. 580 BC–ca. 490 BC), famous for the theorem named after him: that the square of the hypotenuse (longest side) of a right triangle equals the sum of the squares of the other two sides. Pythagoras is said to have discovered the numerical relationship between the length of the strings used in musical instruments and the harmonic combinations of the sounds. In today's language we would describe that relationship by saying that the frequency—the number of vibrations per second—of a string vibrating under fixed tension is inversely proportional to the length of the string. From the practical point of view, this explains why bass guitars must have longer strings than ordinary guitars. Pythagoras probably did not really discover this—he also did not discover the theorem that bears his name—but there is evidence that some relation between string length and pitch was known in his day. If so, one could call that simple mathematical formula the first instance of what we now know as theoretical physics.

So, the tuning of musical instruments is a way of our connecting with one of the oldest and grandest of human endeavors. What we do every day plugs us into traditions that are centuries, perhaps millennia old. Using Math to Model Physics? We do it every day when we manipulate beat rates to achieve our desired tuning.

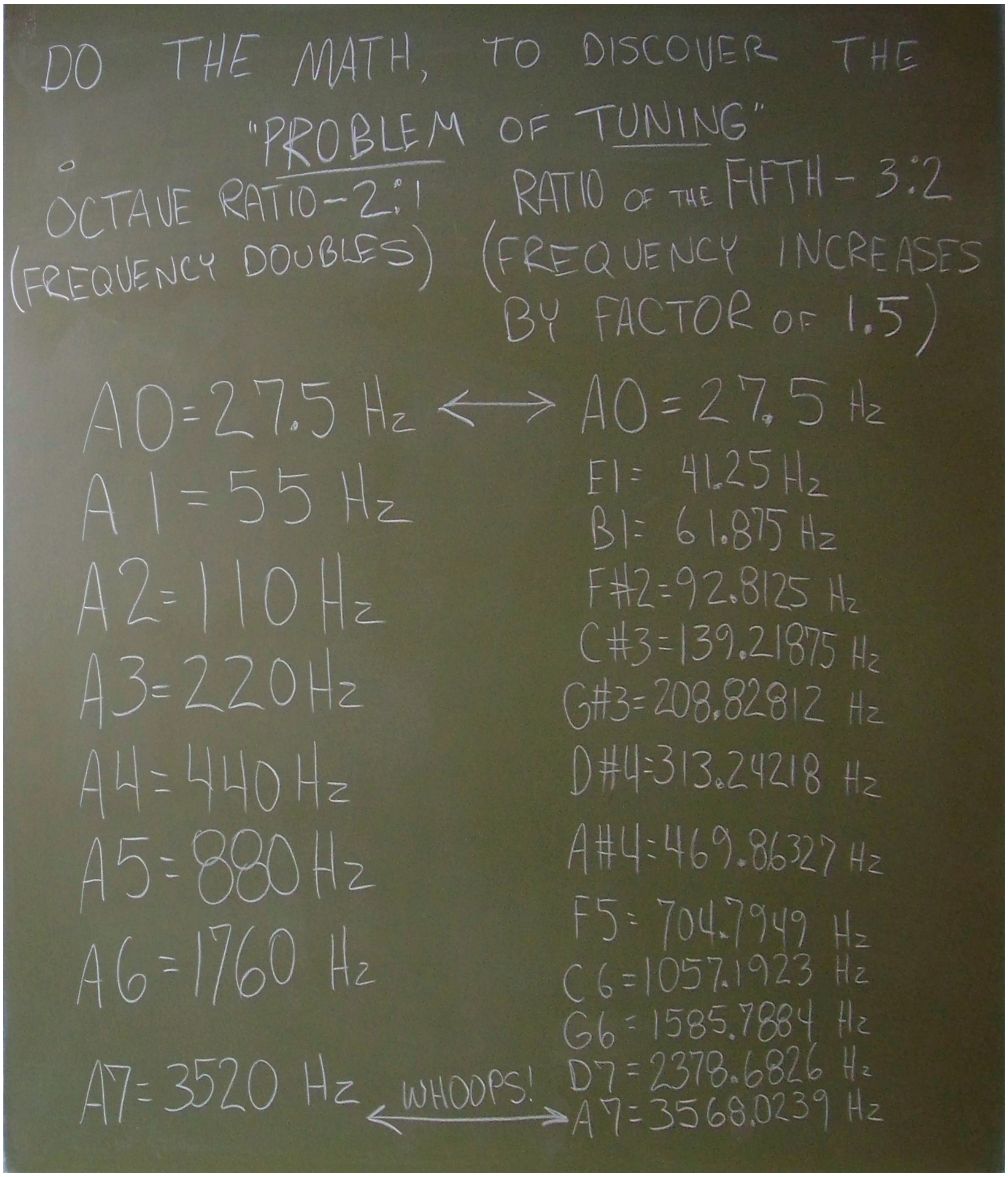
Now, it seems clear that in the beginning the assumption was that *tuning* simply meant pure intervals, that is, all intervals tuned just, pure, beatless. But the problem with all pure intervals soon became apparent.

The Problem of Tuning

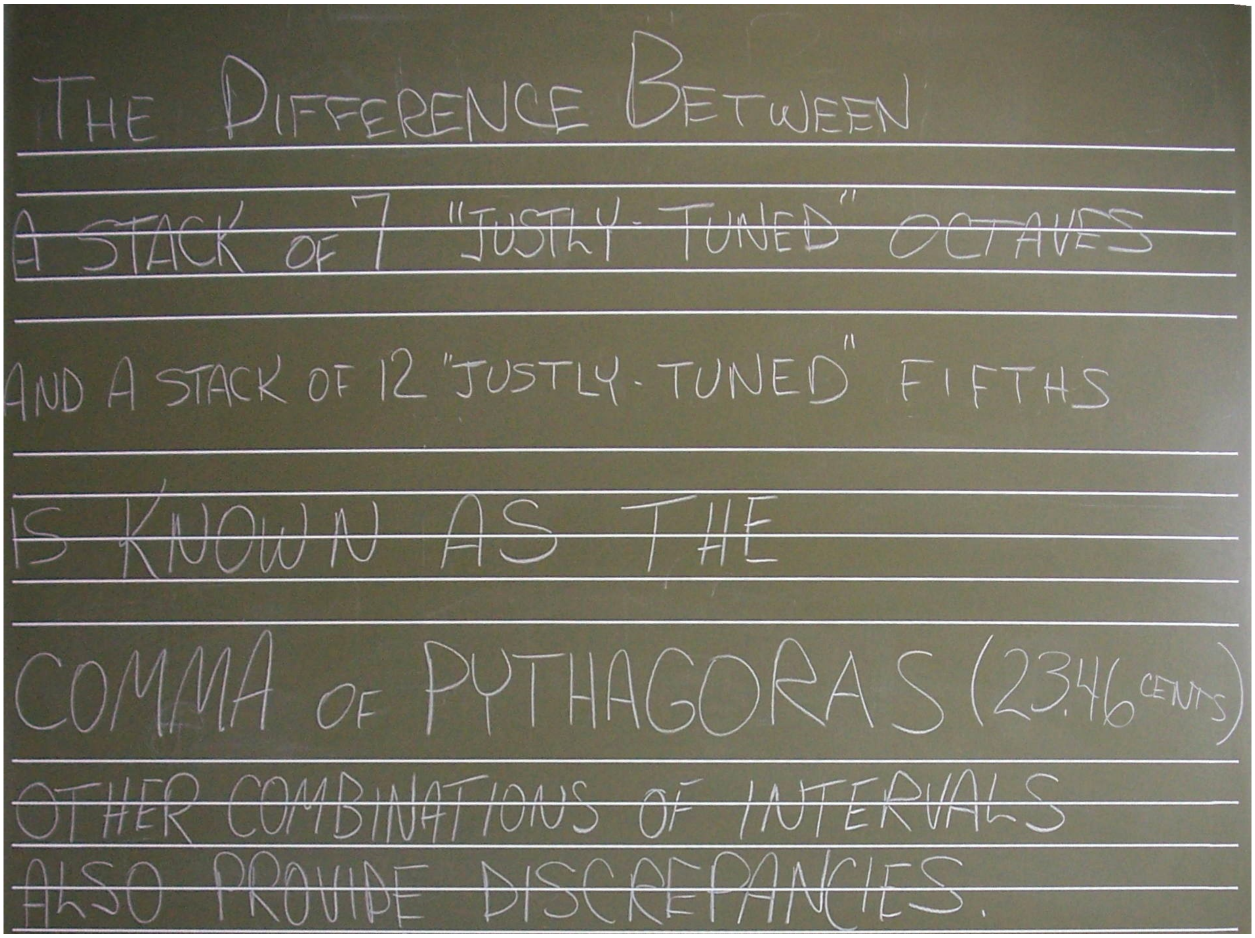
It is said that the contribution of Pythagoras was the discovery that the relationship between just 5ths and just octaves is, uh, problematic, that is, there was no way to tune all intervals pure.

The problem of tuning is that the factors of 2 and the factors of 1.5 can only come together once. If you take a note, say A0, 27.5 Hz, and double the frequency 7 times to form 7 octaves, you will have an A7 at 3520 Hz. If you start at the same A0, at the same 27.5 Hz, and increase the frequency by a factor of 1.5, after going up 12 fifths you come to a note that you might think should also be an A7, but instead of having a frequency of 3520, it has a frequency of 3568.0239

Hz. Whoops. To solve the problem of tuning, you simply must figure out a way for the 2 A7's to match up.



The difference between the 2 A7's is about 48 Hz. This is normally expressed in cents, which is simply a logarithmic way of expressing frequency differences. The Pythagorean comma is about 23.46 cents.



The concept that pure intervals would be the ideal in tuning never really died, particularly with regard to the octave.

“Temperament is the process of altering intervals such as fifths and thirds in order to balance these differences [between 7 pure octaves and 12 pure fifths] while maintaining the octave untouched.” — Fred Sturm

The pure octave remains an ideal to this day. And yet, piano tuners regularly, some would say always, stretch the octave. The reasons for stretching are compelling, and yet the pure octave remains an ideal.

If the octave is sacrosanct, then why do we all stretch them? If you think about it, there is a very real "does not compute" here. If we stretch our octaves then the octave isn't sacrosanct. Period. And there should be no suggestion that the octave must be pure. I suggest that the idea of tempering only the fifths and not the octaves in equal temperament is entirely arbitrary and contrary to what we really do.

Let us explore this together. There is evidence that pure intervals were never the path to the most pure, most consonant *overall* tuning. Perhaps the evidence against pure intervals is compelling enough to label the pure interval ideal as a mistake that needs to be corrected. The mistake or mistakes of Pythagoras may have been that intervals tuned pure or just were ever entirely desirable in the first place, *and* that justly-tuned intervals were a path towards purity and consonance in the whole sound of multiple notes played together. Neither appear to be really true.

There is a very, very big clue in modern pianos with all their inherent inharmonicity. In tuning clean octaves or clean 12ths, the cleanest, purest sound may very well be one in which no pairs of coincident partials at all are exactly pure. This "sweet spot" as we call it is found by listening to the whole sound, or whole soundscape, all at once, without regard to any specific coincident partials. But of course, I am getting way ahead of myself.

To understand all of what I am trying to get at, we need to talk 1) about the history of temperament development. 2) We need to talk about the counter-intuitive ways in which waves combine. And 3) after that introduction, we need to talk about and listen to some of the infinite numbers of modern equal temperaments, and why they exist.

The emphasis in this class is to demonstrate what we can actually hear. We are piano tuners here and we are good at listening and analyzing tunings, so we will do that. Of course, some of these tunings may be different from that to which we are accustomed.

The musical examples here will be computer generated. The sounds you will hear are coming from the computer. The MIDI keyboard is just here as a controller and is not providing any sound at all. The fact that we are not demonstrating with real instruments is regrettable, especially since the computer cannot duplicate the soundboard/bridge coupling effects that are an inherent part of piano tuning, but there are advantages of computerized audio, too. Modern software allows tunings that are very, very precise and completely stable. Often in a classroom full of people simply breathing is enough to take a freshly tuned instrument out of tune, but today in this class, you are all encouraged to breath and breathe deeply, because the tunings will not be affected. And completely different tunings will be available with a tap on the trackpad, so the variety of tunings available during the class is relatively unlimited.

Let's listen to some tuning.

[Audio example of Pythagorean tuning.]

With regard to this tuning, what do you hear? I want to know what you hear. Class?

I have played a musical example designed to show off the "consonant" aspects of the specific tuning.

How shall we evaluate this tuning to discover its overall nature? What aural cues should we be looking for? We take for granted any number of things in the modern world of music: equal temperament, functional harmony based on stacks of thirds, free transposition into any key without changing the character of the music, chromaticism within a given key, enharmonic equivalents. But of course, it wasn't always that way.

I don't know where scales come from, but I know that a perfectly good 12 note scale can be had by tuning only pure fifths and pure octaves. This tuning is an example of Pythagorean tuning. Pythagorean tuning generally refers to stacks of 5ths.

"We shall consider in this [Just Intonation] chapter all 12-note systems that contain some arrangement of pure fifths and major thirds. The Pythagorean tuning may be thought of as the limiting form of just intonation, since it has a great many pure fifths, but no pure major thirds."

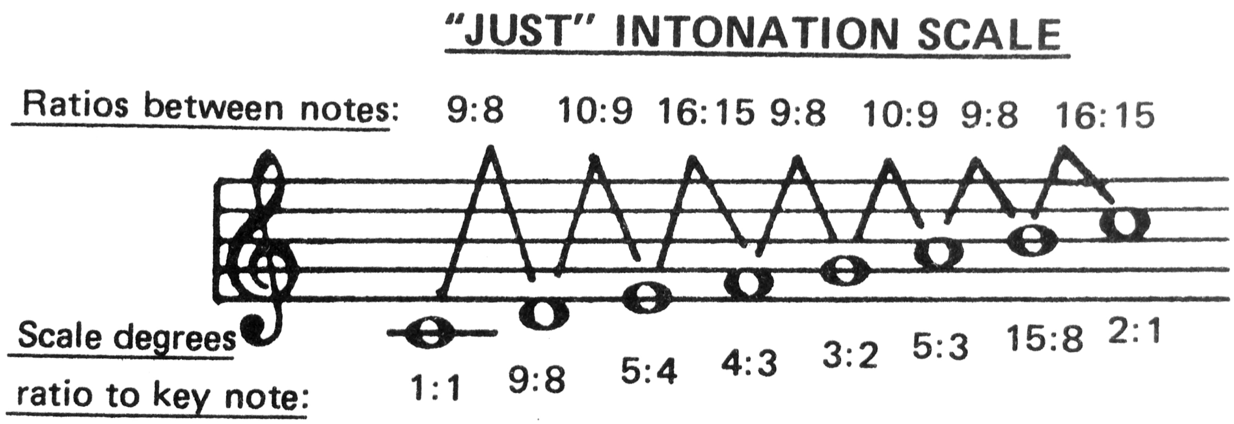
— J. Murray Barbour

In Pythagorean tuning, 11 fifths are pure, but that leaves M3rds quite wide. And that twelfth interval that we might wish to be a fifth is actually an untuned wolf interval.

Playing within the rules of Pythagorean tuning can result in perfectly consonant music. You just have to understand the nature and stay and play within that universe.

Through history, musicians who were unhappy in a given musical universe, simply invented a new one in which to play.

So, when musicians wanted to use the 3rd, they assumed it should be tuned pure, just like 5ths and octaves. And just like today, different musicians came up with different solutions.



[Audio of Example of Just Intonation, Just Twinkle]

Just intonation could have, at least on the C, F, and G chords, pure octaves, pure fifths, and pure 3rds. The problem was that no transposition was possible.

[Audio Example of Just Twinkle in G and F]

So just intonation came to be almost completely displaced by another tuning in development at much the same time.

Quarter Comma Mean Tone Tuning was flat out genius. It took the 4 fifths that in Pythagorean tuning are tuned pure and so leaving the 3rd wide, and narrows them down until this third is pure. So, the octaves are pure, the thirds that are tuned are pure, and the fifths that are tuned are contracted. The tuning has a sweet sound. The fifths are more contracted than in modern equal temperament, but still, the effect can be very pleasing.

A very famous piece of music written by a very famous composer was surely written in the time of quarter-comma mean tone tuning.

Audio Example of Pachelbel's Canon in D, modern piano arrangement but in MT.

Just Intonation was abandoned in favor of Mean Tone tuning because just intonation allowed no transposition and Mean Tone allowed limited transposition.

Demonstrate patterns of 2 tuned 3rds and 1 untuned 3rd that makes up each octave.

There was still a wolf, and this wolf is actually worse than the one in Pythagorean tuning. Limited modulations were possible, perhaps a couple accidentals worth, but the remote keys were quite unusable.

[Audio example of Canon in various keys.]

By today's standards, Quarter Comma Meantone has a huge weakness in its inability to handle much modulation. But it *could* handle modulation by one or two sharps or flats. Just Intonation, on the other hand, could not handle modulation at all, and so Quarter Comma Meantone tuning came into prominence because of its superior ability to use modulation, as much as that sounds strange to us now in the 21st century. Of course, don't feel sorry for Just Intonation because Just Intonation made its own huge comeback in the 19th and 20th centuries, but that is a different class.

To take note, shifting emphases have been in evidence throughout the history of temperament, going repeatedly from one extreme to another.

Just intonation insisted on pure octaves, pure fifths, and pure 3rds.

Pythagorean tunings used pure octaves, and pure fifths, not pure 3rds.

Mean tone tuning insisted upon pure octaves and pure 3rds, but contracted 5ths.

In other words, just about every combination, except tempered octaves. Why? Well, that is what this class is about! But, back to history for a moment:

Musicians worked within a given tuning system's limitations — except when they didn't! Musicians understood the rules, but pushed the boundaries when they needed.

Over time remote _keys_ were desired _and_ remote, more adventurous harmonies were desired within each key.

The well and good temperaments insisted upon pure octaves, but had a mix of pure and contracted 5ths, and a mix of expanded thirds. The octaves were pure, and in the keys close to C major the thirds were close to pure. Musicians had developed the fine art of tempering, which is the art of carefully controlling the out-of-tuneness of intervals to achieve a desired musical effect. A case could be made that the Well and Good Temperaments are the most sophisticated temperaments, even more so that equal, but that also is a different class. Let us listen to a temperament for what it can teach us about analyzing temperaments aurally.

Demonstrate Shifted Valotti / Thomas Young No. 2

From C, descending 5ths are pure. From C, ascending 5ths are tempered by about twice that of standard equal temperament. This combination graduates the M3rds as you move away from C in either direction through the circle of 5ths. This is a brilliantly ingenious temperament, and it is relatively easy to tune, even aurally, and should be recognizable because the beat rate patterns are so straightforward. And parenthetically, even if you are tuning this temperament visually with an ETD, it would be simple to check the temperament aurally to see that the pattern of beat rates is actually present. To re-state that opportunistically as an slightly off-topic opinion, we should learn the characteristic beat rate patterns of the temperaments we are tuning, even when we are tuning with an ETD, so that we can make sure that we have chosen the correct temperament from the ETD menu, so that we can make aural corrections as necessary, so that we can demonstrate the tuning for the customer, and the list goes on. And this applies to historical temperaments and especially to the modern equal temperaments that we will be discussing shortly.

And While We Are Off-Topic:

Shifted Valotti/Young #2 is not Equal Temperament. But looking at well and good temperaments from the point of view of Mean Tone tuning, you could use all the keys, so would it really be so far-fetched if a temperament such as this had been represented as equal temperament? So mistaking well and good temperaments for equal temperament may have happened early on. After all, they had no way of knowing how accurately future tuners would learn to tune. They may very well have thought good and well temperaments were equal temperament. I think it may have been equal enough to enable the development of chromaticism.

Rhythm example in mean tone
Rhythm example in well
Rhythm example in equal

Getting back on topic, think back on the development of temperament. Quarter-comma meantone, fifth-comma meantone, sixth-comma meantone, seventh comma meantone, eigth-comma mean tone — does one begin to see a pattern here? The development wasn't neat and orderly and linear, but the comma came to be split up between more and more of the scale tones. When fifths were contracted by 1/12th of the Pythagorean comma traditional equal temperament resulted. Equal temperament is generally not considered to be a mean tone tuning because there

is no wolf, but equal temperament could certainly be seen that way given the progression of temperaments that came into being through the centuries. They then continued to divide up the comma into more and more pieces beyond 12, as we will explore shortly.
Now, we have to talk just a moment about frequencies and beats on the one hand and cents on the other. If you learned to tune aurally, you know that cents are rather foreign and to aural tuners cents are associated with electronic tuning. But, which came first, cents or electronic tuning? The answer of course is cents came first and were adopted by electronic tuning devices when they came along, out of convenience. So, what was this convenience? Cents are logarithmic expressions of frequency differences. But the important thing is that cents make calculations that would otherwise be fiendishly difficult with frequencies into dead simple calculations in cents.

Calculate the frequency difference of the Pythagorean comma and convert to cents. In cents, one can manipulate the Pythagorean comma with simple arithmetic and algebra.

Remember, we said the Pythagorean comma is 23.46 cents. So to divide the comma into 12 parts we simple divide 23.46 by 12. The answer is 1.96, or approximately 2 cents. Heard of that before? Contract the fifths by 2 cents and tune pure octaves, and the circle of 5ths will close out perfectly, and the comma is vanquished. Well, perhaps vanquished is too strong a word.

We know the beat rate relationships of 12 tone to the pure octave equal temperament, right? Intervals progress smoothly. The octaves are pure; the fifths are two cents contracted, as shown by the 6th,10th test; in the 4th-5th test of the 4:2 octave the 4th and 5th beat the same. It has been tuned aurally for a long time.

There have been alternative ways of dividing up the comma, even among those wishing to tune equal temperament.

When we talk of contracting the 5ths by 2 cents and leaving the octave pure, as it turns out, that is only the narrow end of equal temperament. There is some indication of this in the literature. For example, here is Daniel Levitan:

“First we might tune notes from F4 to F5. The range of acceptable tuning for these notes is generally considered to be no narrower than a pure 2:1 octave, and no wider than a pure 3:2 P5.”

Introduction to Pure 5th Equal Temperament [Play example of pure 5th ET]

An alternative equal temperament in which the fifths are left pure and the octaves expanded does exist. While pure 5th equal temperament was not clearly described and codified until 1959, there are claims that it had been in use as early as 1809 and passed down aurally tuner to tuner.

Let’s use our knowledge of cents to figure out pure 5th equal temperament. The Pythagorean comma is still 23.46 cents, but where there are 12 fifths in the circle of fifths there are just 7 octaves, so the comma must be divided into 7 to achieve pure 5ths. Doing the math, the octaves must be expanded by 3.35 cents.

So, traditionally, temperament is dividing up the comma, leaving the octave alone, but here the 5th is left alone.

This seems like it might have been a fairly radical idea. Given that the octave was traditionally off limits for tempering, and here it is tempered by 3.35 cents, this might be considered an extreme temperament. However, if pure 5th equal temperament is extreme on the wide side, then pure octave equal temperament may be extreme on the narrow side. And I should suggest that our propensity for stretching the octave and tuning wider than pure octave equal temperament tends to bear this out!

Perhaps the useful range of the widths of equal temperament should be considered to be no more narrow than pure octave 12th root of 2, and generally no wider than the pure fifth 7th root of 1.5.

Let us take a moment to examine some of the beat rate relationships in this pure 5th equal temperament. Intervals progress smoothly. Here are the sound of pure 5ths. Here are the sound of octaves that are expanded by 3.35 cents. A little much? Here are the seventeenths. What about the 4ths? If the 5ths are pure what do the 4ths sound like? They are expanded by the same 3.35 cents as octaves!

Do you get the idea that this could be tuned aurally? Just as pure octave equal temperament is tuned aurally, so too can pure 5th equal temperament be tuned aurally. Smoothly progressing intervals, but arranged so that 5ths are pure, and 4ths and octaves are both tempered the same. This fourth (C3-F3) coincident at 4:3 on C5 would beat the same as this octave (C4-C5) which us coincident at 2:1 also on C5.

Introduction to Pure 12th Equal Temperament

But, as piano tuners, we like to have things both ways, right? 8^) We like to think we are masters of compromise. Could there be an equal temperament which is a middle ground between pure 5th equal temperament and a pure octave equal temperament? One in which the fifths and octaves are tempered by the same amount? This is where we can use our math and figure out the cents involved in such a temperament.

We want to temper 12 5ths and 7 octaves by a single amount. So, we divide the amount of the comma by 12+7, that is, 19. That comes out to about 1 1/4 cents. Can it really by that simple? Actually, yes, it can be and it is that simple. If we contract the 5ths by 1 1/4 cents and expand the octaves by the same 1 1/4 cents we can build an equal temperament mid-way between pure octave ET and pure 5th ET.

Note that if the octave is expanded by the same amount the 5th is contracted, and the two are stacked, you get a pure 12th. This temperament is pure 12th equal temperament; you may have heard of it. 8^)

Instead of 2 cent fifths we have 1 1/4 cent 5ths and instead of pure octaves we have 1 1/4 cent octaves. This can give a *very* clean sound when three or more notes are sounding.

Indeed, as Fred Sturm says,

“Fifths narrow by two cents are hardly noticeable in a musical context, and that small change is difficult for most people, including musicians, to hear.”

That comment refers to traditional equal temperament and yet this pure 12th temperament is a temperament that contracts the 5ths even less.

I can hear someone thinking, “Yes, but the M3rds will be too fast.”

But according to Di Veroli:

“[In usual octave stretching], where they matter, in the keyboard’s midrange and treble… the octaves are stretched by about 1 Cent and therefore the Equal Tempered Major Thirds are widened by an imperceptible 0.3 Cent only.”

Di Veroli is referring to standard equal temperament. But if we widen the octave stretch from 1 cent to 1 1/4 cent, the M3rds will be additionally widened by less than 0.4 cents. As tuners, we may very well hear this, but, is it really enough faster enough to be unacceptable?

And one more thing: If musicians cannot distinguish a 2 cent 5th from pure, what makes anyone think that they could distinguish a one and a quarter cent octave from pure?! Just saying.

Let us listen to some of the beat rate relationships. Again, this is equal temperament so there are smoothly progressing intervals. The 12ths are pure. And the 5ths and octaves are tempered by the same 1.24 cents. This means that this (C3-G3) 5th beats the same as this (G3-G4) octave. Both the 3:2 fifth and the 2:1 octave have coincident partials at C5; the tempering is the same so the beating is the same. There is more; this 4th, (C3-F3) coincident at 4:3 on C5, and this double octave, (C3-C5) coincident at 4:1 on C5 will beat the same. And what about the 4ths and 5ths? Taking the notes of the 4:2 octave test, in a zero inharmonicity situation, the 4ths will beat twice that of the 5ths. All coincidences are at C5, this 4th is expanded by 2.48 cents, and this 5th will be expanded by half that, 1.24 cents! Again, this is enough to tune this aurally.

Theoretically, dividing the Pythagorean comma into 19 parts is no less arbitrary than dividing it into the 12 parts of traditional equal temperament. Which is just to say that one could divide it into other number of equal parts as well. The CHAS tuning from Alfredo Capurso in Italy comes very close to equally tempering the 12th and double octave, which results in a pure triple octave 5th. Yes, this divides the Pythagorean comma into 43 parts, making it only the slightest bit wider than traditional equal temperament, representing it mathematically with the 43rd root of 12. In fairness, Capurso himself describes his temperament rather differently than how I just did, but my description is simpler! 8^)

One could also equally temper the 12ths and octaves, making a pure 19th (double-octave 5th), representing it mathematically with the 31st root of 6, dividing the Pythagorean comma into 31 parts.

So, historically there have always been a full range of temperaments.

For example, there are temperaments with pure thirds and Pythagorean 3rds and everything in between.

There are temperaments with pure 5ths and ones with quarter comma contracted 5ths, and everything in between.

Well, there have been temperaments with pure octaves and temperaments with octaves expanded by 3 1/3 cents. So, I am suggesting octave widths of "everything in between" just as it has been with other intervals through history. OK, enough history.

When Waves Combine

Let us talk a bit about combining waves. What happens when waves combine? It is interesting.

Thomas Edison is said to have first demonstrated that the waves of multiple sound sources can be accurately represented by merging the separate waves into a single complex wave. Were this not so, the phonograph, and indeed any audio recording at all, could not exist. So, for one thing, we know that waves combine to form more and more complex single waves. One wave + one wave + any number of more waves, equals one wave, albeit one very complex wave.

Boat Video

In this video you see the boat subjected to the same wave, but in one, ballast tanks are allowed to take on and let out water, providing a second wave. Note the rocking on the boat on the right, but it is actually the left boat going up and down almost perfectly without rocking which is exhibiting the more interesting, more complex wave. So two waves could conceivably combine to result in less motion than one wave alone?

You can see that the combined effect here is rather different than either wave by itself. You must look at the whole situation. In piano tuning, I daresay, one must listen to the whole sound to use Virgil's term, or the whole soundscape to use Brian Capleton's term.

Coupling — Metronome Video

Coupling obviously occurs and is not particularly mysterious. In pianos, it is simply the result of vibrating strings not being fully terminated and allowed to influence each other.

Coupling — Unisons, Capleton

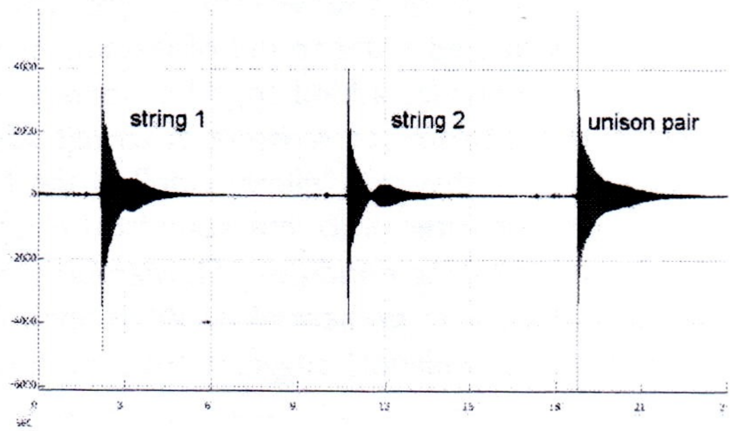
This has an obvious application to tuning unisons on the piano. Here is the graph of an example of two different strings of a unison sounded individually and then sounded together. One might expect that the two waves get simply mathematically added together, but that is not what happens, the sound of the two together is different than either one by themselves might lead one to expect. While each individual string has its own false beat when sounded by itself, the two strings can be tuned together to produce a good beatless unison.

Examples of false beat attenuation

Example 1 – Steinway A49 (A4)

Fig. (8.13) illustrates one example of false beat attenuation in the second partial of a two string unison A49 (A4) of the Steinway model M, the third string being wedged.

Fig. 8.13



In fact, two slightly mistuned vibrating strings may not create a beat. It just is not that simple. Between falseness and coupling, the frequency of a unison may be somewhat different than the frequencies of the individual strings of that same unison when sounding by themselves.

Beats arise as frequency differences between vibrating strings. However, we *hear* them because of amplitude modulation, that is, the rising and falling of volume. Amplitude modulation can clearly be used to attenuate (mask) beats. Therefore, as counterintuitive as this may be, past a certain point, even though beats arise out of frequency differences, frequency doesn't matter in quieting beats. What does matter is the combined amplitude envelope of the (perhaps coupled) group of strings and whether the decay of that amplitude envelope is smooth or not. Since unison tuning is subject to random decay envelopes, and two or more of them may couple to create a combined decay envelope different from any of the individual string decay envelopes, this means that unison tuning itself has a random element, that is, trial and error may be the only way to tune unisons, and in a given unison there may be more than one solution to achieve a pure unison.

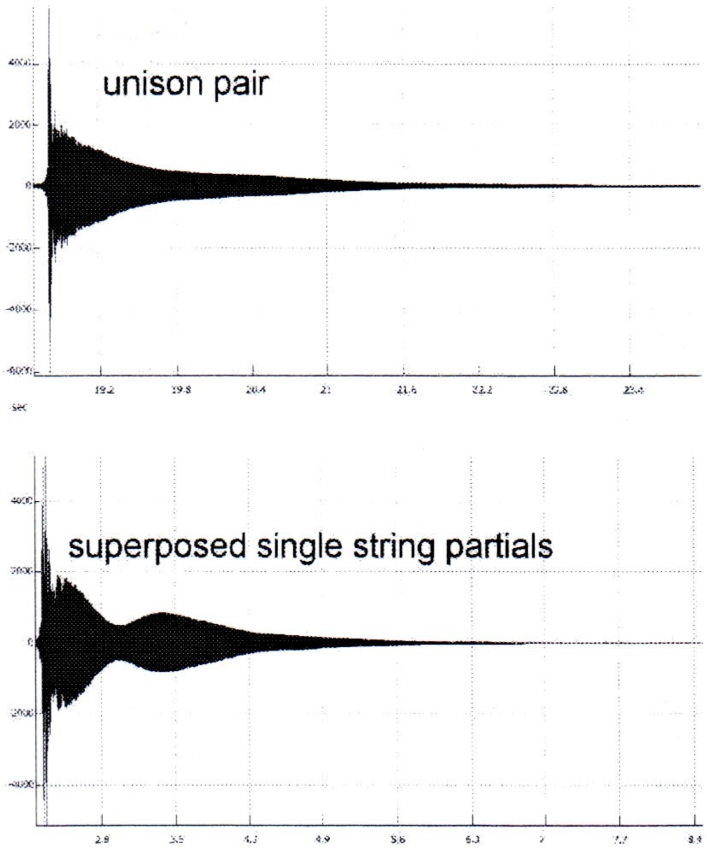


Fig. 8.13. Envelopes of the second partial for each of two strings of the A49 (A4) unison on a Steinway model M, and of the unison pair. The actual envelope for the unison pair is also shown (middle) magnified. At the bottom is the envelope resulting from the digital addition of the envelopes for the single strings. In terms of reducing beats or beat-like movement in the partial, a better result is attained in the actual situation where both strings are sounding, than can be synthesised by simply adding the two envelopes from the single strings sounding alone.

"Once both strings are sounding together, the effects of the bridge and soundboard coupling the two strings can mean that the actual frequencies produced are no longer as they would have been, had the strings been sounding separately."

-- Brian Capleton

I believe this begins to explain why student tuners obsess over false beats and experienced tuners don't even hear them. Over time, tuners may learn how to eliminate many false beats with clever tuning! Which may mean that, to an experienced tuner, an audible false beat doesn't register as a false beat but rather just means that he is not done tuning the unison!

"Fine unison tuning is not generally carried out with conscious regard for movement in this or that particular partial, but with regard for movement in the soundscape as a whole, in the context of the tone quality of the soundscape as a whole."

-- Brian Capleton

"In general, unisons are not tuned by concentrating on specific partial decays. In fact, concentrating on any one partial is a recipe for poor tuning. The unison as a whole is like a whole painting -- we must 'step back' aurally and observe the whole, which means to listen to the whole without concentrating on any particular partial. This does not necessarily mean we do not hear particular partials."

-- Brian Capleton

As for myself, I tune all 3 strings of high treble unisons with an ETD. Because random tuning differences can sometimes sound clean, I want to be as close as possible first and introduce tiny random changes as necessary to get the best unisons possible.

So I hope what we are beginning to see here is that waves combine in unexpected, non-intuitive ways. And we might not know everything that there is to know about how waves combine.

Synchronous Temperaments

Just a brief word about synchronous temperaments. Synchronous temperaments are those temperaments in which beat rates have whole number relationships to each other like 2:1.

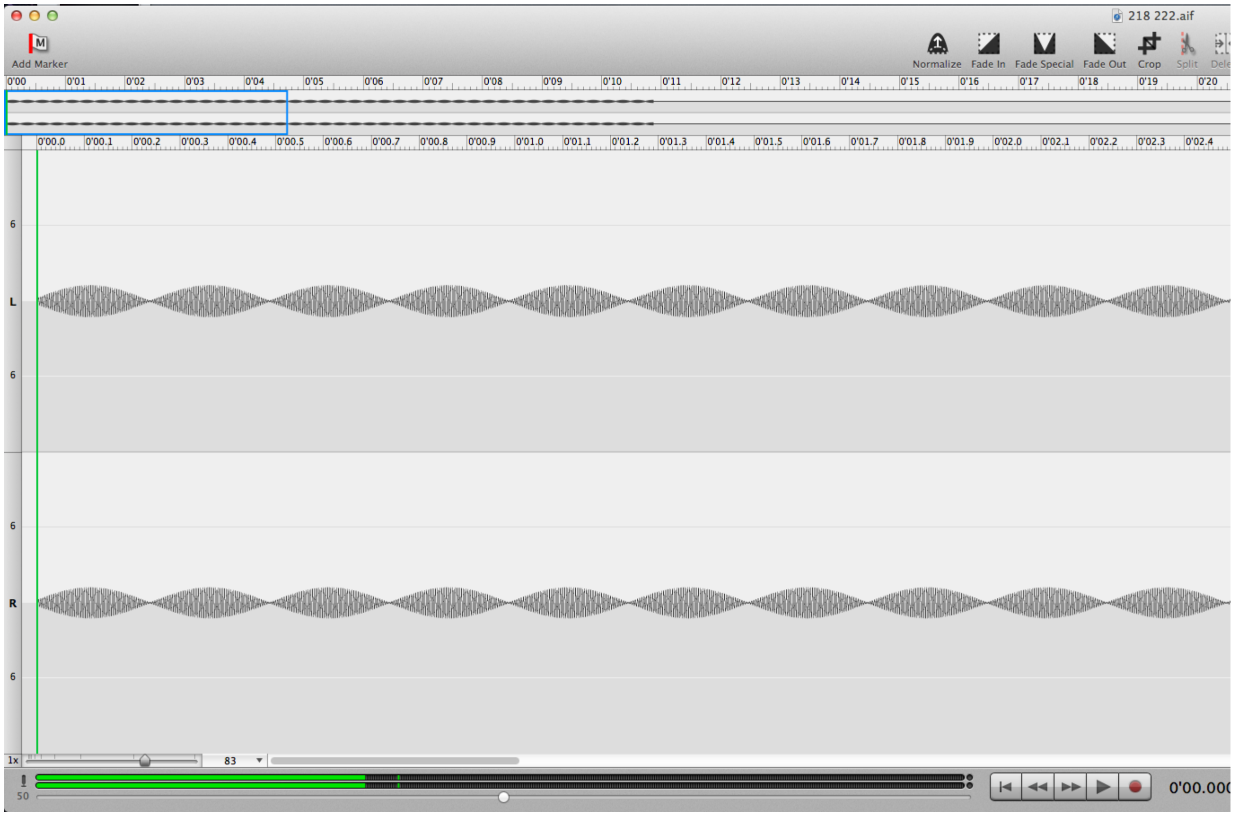
Don't confuse synchronous tuning with the pure sound of wide equal temperament. Synchronous tuning is not about overall purity, but rather about a coherent, clear vibrato effect. Synchronous beat rates do not cancel each other, but rather, they tend to accentuate each other. Chaos is reduced but purity is not increased; it just makes the vibrato more clear.

Beat masking as in the whole sound of wide equal temperaments tends to take place among beat rates that are *asynchronous*. So, as an example, tuning out a false beat is not about matching the false beat rate and canceling it out, but rather finding a random asynchronous beat rate that happens to mask the false beat.

The Truth About Beats

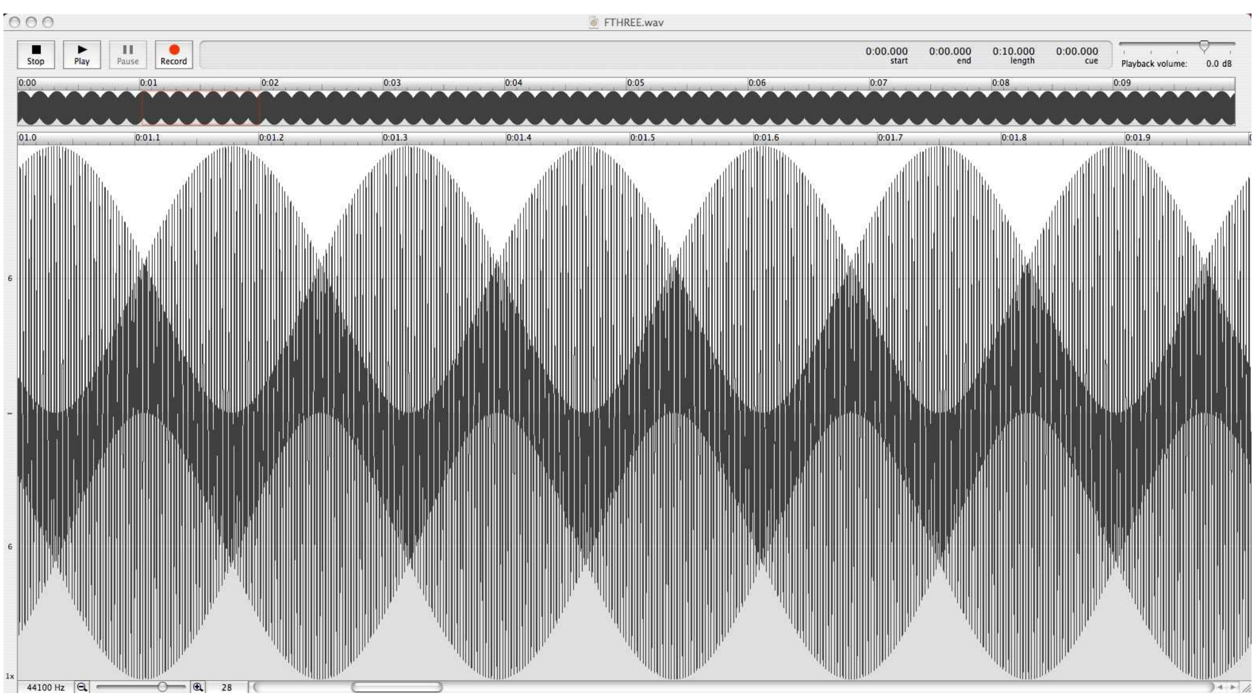
Audio example of two sine waves, 4 beats apart, with clearly defined beat.

Coincident partials form beats when they are only a few cycles per second apart. True.



Audio example, two sine waves that demonstrate a simple beat, not as pronounced but clearly present.

So is it true or false that all beats arise from coincident partials?



What do you hear in this second example? Certainly not beats that are as prominent as the previous example, but what do you hear? What you are listening to are two sine waves one corresponding to F3 and the other corresponding to A4 in traditional equal temperament. These are sine waves, there are no partials, only fundamentals. Here is the sound wave loaded into an audio processor application, and as you can see, the beating develops between the fundamentals alone, and while completely in the digital domain.

To those who would say the beats are the result of distortion of the computer sound as it is transformed from computer math to real world sound, I would just point out that beats appear in the graphic representations of the sine waves alone even though no partials at all are represented in the graphs.

In looking at the graphic representation of the beating between 2 fundamentals, no particular proof of the lack of harmonics is required, because one can see in the graphs directly that the beats form from the combination of two fundamental waveform without the presence of any harmonics at all.

It is just math; if you draw the two fundamentals alone and interacting between the two, beats happen. No partials are present in the graph, but beats are present as a result of the 2 fundamentals interacting.

So is it true or false that all beats arise from coincident partials? It is absolutely false, and easily demonstrable, with simple theory to back it up. Wait. Simple theory?

The theory behind beats without partials has been staring us in the face all along, because we already know the theory. The theory behind this phenomenon of beats forming between fundamentals is the same theory as for beats themselves. We must remember that beats are sub-sonic difference frequencies; and difference frequencies that are fast enough to be heard by human hearing as pitched tones are called difference tones, and difference tones and beats are most definitely the same phenomena. And it is difference frequencies that ultimately explain the presence of beats in effect forming between fundamentals.

Two audio frequencies when combined produce the original frequencies plus a new frequency at the difference in frequencies between the two original waves. We all know that 2 tones, one at 440Hz and one at 442Hz will create a beat, or difference frequency, of 2 Hz. But 2 tones, one at 440 and one at 174.61 will also create a difference frequency, but one that is itself in the audible range, and could possibly be heard as a new tone. Difference tones are just as real and just as potentially audible as beats.

Take the present example of two sine waves, one at F3 and the other at A4:

The frequency of A4 is 440 Hz.
The frequency of F3 is 174.61 Hz
(A4 = 440 12- tone to the pure octave ET, zero inharmonicity)

440 - 174.61 = 265.39

265.39 - 174.61 = 90.78
174.61 - 90.78 = 83.83
90.78 - 83.83 = 6.95

Eventually, you reach a difference frequency that falls below the range within which humans can hear as a tone; but we can still hear it as a pulsing — as a beat.

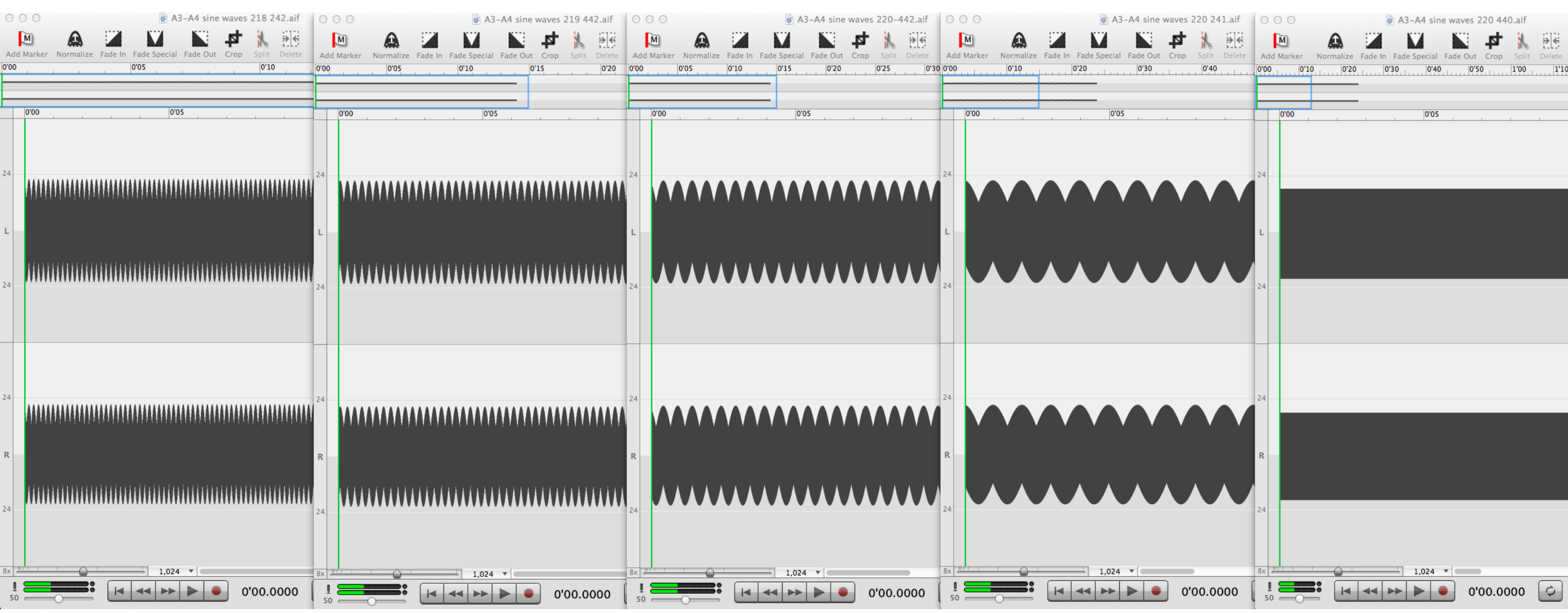
Combining waves at the fundamentals can produce beats. Virgil Smith claimed as much and was brutally criticized for it; may I modestly suggest that these files, which I produced, by the way, in 2006, show that Virgil may very well have been correct!?

Daniel Levitan writes in his book "...we cannot tune the 1st partial of A3 directly to the 1st partial of A4…" There are, after all, no coincident partials just a few beats apart. But now we know that coincident partials are not the only way to form beats! And now we know how to properly extend the theory to fundamentals as we should. Here is how it should go:

In the example of two sine waves, one at 442, and one at 218, there would be a difference tone at 442-218 HZ, or 224 Hz, which is a real tone, same as any other. Then the difference tone at 224 would beat with the original tone at 218, and would beat 224-218=6 bsp. Coincident partials can beat, and coincident difference tones also beat.

Audio example of decreasing beat speeds. You'll hear a few seconds of one beat rate, then a pop after which the beat rate will have changed.

Graph of decreasing beat speeds. A3-A4 Sine waves with decreasing beat rates.



The tones are sine waves, that is, fundamentals *only* at A3 and A4. As we move the tones closer to the 2:1 frequency ratio, there are beat rates and they diminish. May I suggest that we just tuned the first partial of A3 directly to the 1st partial of A4!?

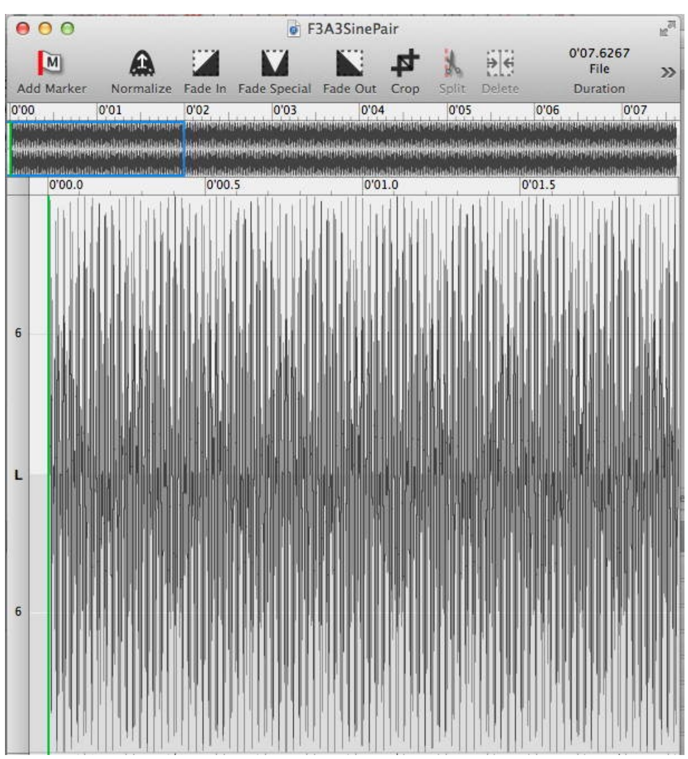
Now, I did not just prove Levitan wrong. He is right if you just say, you cannot tune A3-A4 *on a piano* with fundamentals alone. But think about it, we just demonstrated the complexity of two sine waves an octave apart combining. This very same complexity happens on the piano; there are just other things, partials, happening at the same time. Do you get the idea that combining waves might be a complex thing? We may not know all that there is to know about combining waves.

Let's be clear about the theory of beats between fundamentals. If a theory is correct it can make predictions, so let's make one.

Let's consider two sine waves forming a M3rd.
One tone is F3 at 174.61Hz.
One tone is A3 at 220Hz.
What will happen?
A difference tone will form at 220Hz - 174.61Hz = 45.39Hz
A difference tone should form at 174.61Hz - 45.39Hz = 129.22Hz
A difference tone should form at 129.22Hz - 45.39Hz = 83.83Hz
A difference tone should form at 83.83Hz - 45.39Hz = 38.44Hz
A difference frequency or beat should form at 45.39Hz - 38.44Hz = 6.95Hz
If there is a beat of 6.95Hz, it is likely to be a very weak one.
A number of audible difference tones will be present.

Let us listen to the actual wave. [Audio Example F3-A3 M3rd sine wave pair.]

Take a break and ponder!



Modern Equal Temperaments

How many piano techs does it take to change a light bulb? Just one, but all the other piano techs will be watching, saying, "You know, he's doing it all wrong!"

21st century piano techs have resources and tools undreamed of in centuries past. Wendy Carlos, the electronic musician says, "Any parameter that *can* be controlled, *must* be controlled." Is that true for piano techs and tuning? Because of all the knowledge that is out there, must we control our tunings in ways that our teachers never taught us? I am beginning to think so!

Like all the questions I am raising today, I don't have the answer, but I sure do think that these are some interesting questions.

We are piano tuners, and we like the idea of compromise. We fancy ourselves masters of compromise. Well, at least a piano tuner likes his own compromises. I find that no one tends to like a tuning compromise unless he is the one who made it. Just remember that that finely wrought, exquisitely balanced tuning compromise that you expertly executed, to other techs, just sounds like a mistake!

But no matter, we like compromise, and there exists a whole world of equal temperaments that divide up the comma by tempering both the fifth and the octave.

First, just a few comments about our disagreements as piano techs. I see the same thing over and over. We are in substantial agreement about many, perhaps most things, but we spend a lot of time arguing anyway, often over the tiniest, perhaps unimportant details.

It is my opinion that we should make the effort, when someone says something we disagree with, to discover under what conditions the other tech's statement might be true, not the conditions in which we know the statements might be untrue. Like I did with Levitan's statements that you cannot tune an octave using fundamentals.

Disagreements come up all the time, sometimes seemingly irreconcilable. Here are two quotes about difference tones, which are beats with frequencies in the audible range.

"For the bass octave, the simple difference tone may be significant."

-- Brian Capleton

"Regrettably, as interesting as the phenomenon of difference tones is, as far as I know, it is not of much use in piano tuning."

-- Daniel Levitan

I cannot resolve these two statements, but it is my guess there are conditions under which both can be right. We should look for those conditions as a courtesy that should be extended to everyone, just like a golden rule:

Find the truth in other's statements just as you would have them find the truth in your own statements. This is how we can grow and broaden the group's body of knowledge, to incorporate others' experience into our own.

We owe it to people like Virgil Smith. If people had worked with Virgil to discover the ways in which he was right, piano tuning might be more advanced today than it is. I have to say that the excoriation that Virgil Smith endured was so strong that even today I am afraid of defending him for fear that people will make me out for a crackpot by association, but for all that Virgil got wrong, he got some very important things right, so here I am today to say, "Virgil Smith made a real contribution to what we do." There may have been a time when I was one of the ones dismissing Virgil; it gives me great pleasure to stand here today and correct my mistake.

We can choose to dwell on disagreement, or we can choose to emphasize those things about which we agree. Let me give you an example.

"In any case, since, because of inharmonicity, not only are virtually no fifths or twelfths pure in any piano, but also their sounds vary not only from piano to piano but also from register to register within a piano, we should be highly skeptical about all claims for the special nature of any piano tuning system based on the pure tuning of these (or any other) intervals."

-- Daniel Levitan

Well, you might think that since I went to the trouble of introducing the pure 12th equal temperament in the previous class period that I might be upset by Daniel's statement, but I am not upset in the least. He's right. At least we agree about the futility of trying to tune pure intervals on the piano! I also pointed out in the first class period that in tuning clean octaves or clean 12ths, the cleanest, purest sound may very well be one in which no pairs of coincident partials at all are exactly pure.

Think about it! If pure intervals are not possible, then pure tones cannot be the path to the best, most coherent tunings, simply because there is no such thing as pure intervals on the piano. So the question is always which combination of tempered intervals gives the most coherent tuning?

I would suggest that the most coherent, pure-sounding tunings come from executing as accurately and consistently and evenly as is possible a single stretch level across the whole scale of a piano. Now, give me the rest of the class time to explain that.

So, this class is an exploration of some options in equal temperament that perhaps we were unaware existed.

Here is a quote from Bill Garlick:

"In equal temperament on the modern piano there is not a single interval which is tuned just or perfect -- even including the octave. Due to the Comma of Pythagoras and inharmonicity all intervals must be contracted or expanded from perfect and will therefore beat... All the tempered intervals of equal temperament should gradually increase in beat speed evenly, ascending chromatically. It should be noted that such chromatically ascending progressions, particularly of M3rds and M6ths is a characteristic unique to equal temperament and distinguishes it from any other temperament."

That definition, which I think is brilliant, inherently allows differing levels of stretch. Evenly increasing beat speeds is the only requirement; it doesn't say anything about absolute beat rates, only smooth relative beat rates of ascending intervals. So, the amount we stretch varies. But should it vary, especially within a single tuning?

De Veroli suggests that common practice is to stretch 1 cent per octave in the mid-range.

OK, what if we stretch more than that — or less than that? Either way, it can still be equal temperament, and either way, it may very well not be 12th root of 2 equal temperament.

[Demonstrate cup holder as a model of ET.]

Imagine an expandable cup holder as a model of equal temperament. No matter the exact expansion, the joints are evenly spaced, equally tempered, if you will.

Visualizing the expanding cupholder as a model for equal temperament, can you see that the exact level of expansion only needs two points to be identified.

This means, then, that the width of your equal temperament in a tuning is determined by the second note you tune, whichever note it may be.

After tuning your second note, every other note's position is written stone if your temperament is to be equal.

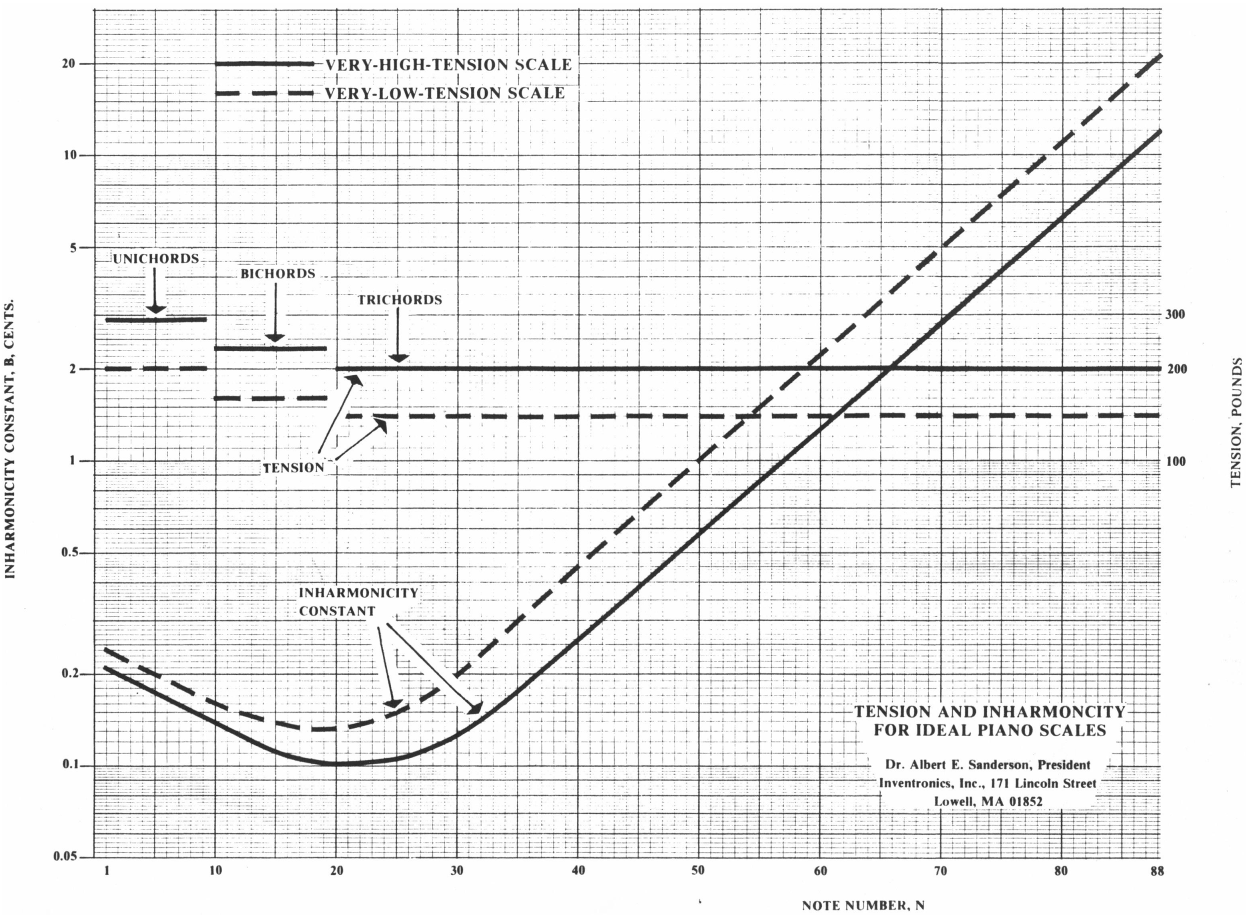
OK, so there can be narrow equal temperament and wide equal temperament — so what? What difference does it make?

Well, in that cup holder, you cannot easily combine widths; only one width at a time is possible. With a cup holder, if you try two different widths at one time, you will break something. With tuning, if you have more than one width of equal temperament at one time, then you will no longer have equal temperament. The equal temperament will be broken!

We must talk more completely about stretch, and then relate it back to equal temperament. Here goes:

Why do we stretch? I would like to give the class the opportunity to comment. What are your ideas? I am not looking for any one answer that I think might be correct. I wish to know what the general understanding is of why we stretch. Anyone?

Pianos are scaled in such a way that their tunings are capable of emulating the beat rates of zero inharmonicity tunings. This was taught by Al Sanderson. He would display his graphic depiction of the perfectly scaled piano, here it is, and point out that pianos certainly could be scaled so that it was impossible to emulate the theoretical beat rates, but instead they were scaled to allow an approximation of the no-inharmonicity 12th root of 2 equal temperament. And if they were scaled ideally, then octaves could be tuned perfectly, because the increase in inharmonicity as you ascend through the scale would match the increase in inharmonicity as you ascend up through the partial series of individual notes.

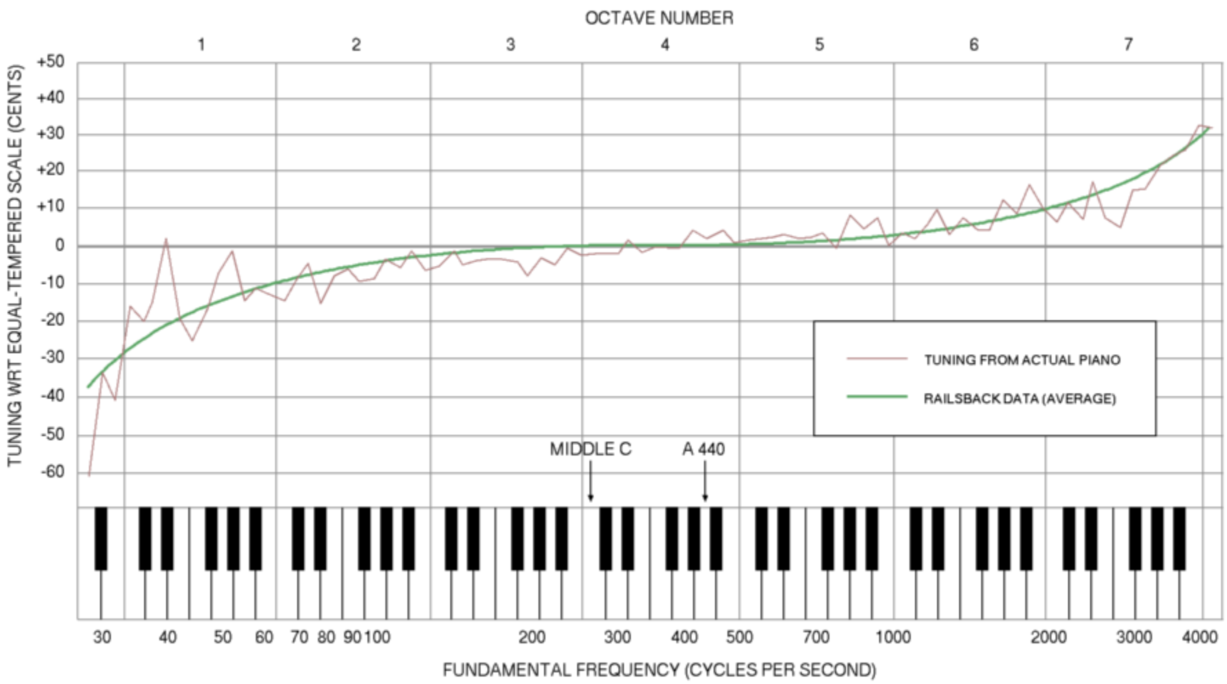


As Daniel Levitan puts it: "...Piano tuners always prefer to approximate to some extent the theoretical beat rates of equal temperament, purposefully mistuning 1st partials... This is because piano tuners listen, not to 1st partials, but to coincident partials. Through the artful mistuning of 1st partials, tuners strive to create the illusion that there is no inharmonicity in the intervals of a piano."

And that mis-tuning of fundamentals as Levitan aptly puts it, is stretch of a sort, because it widens out the temperament to fit into the wideness demanded by inharmonicity. As Daniel Levitan puts it, "Widening the temperament octave restores the temperament intervals to their theoretical beat rates, each at a different point in the expansion." However, for the purposes of this class, stretch involves tempering the octave. If partials are inharmonic, that is, if they are sharp because of inharmonicity, and if you tune beatless to those sharp partials, the resulting interval is not stretched, it is simply corrected for inharmonicity. **(This means that the wideness**

of the equal temperament is determined by how much the octave is tempered beyond that needed to correct for inharmonicity. This wideness can be measured directly or determined by analyzing the beat rate relationships.)
Matching inharmonicity is one thing, but stretching beyond inharmonicity is another thing, and we all do that. De Veroli suggests we temper the octave 1 cent in the mid-range. Why do we do *that*?

Again, Dr. Al Sanderson provided an answer decades ago. I found class notes where he taught the Railsback curve. For some reason we don't seem to talk too much about the Railsback Curve in PTG; we let the non-Guild guys talk about it at this point. But it is a great concept.



This is an aural piano tuning graphed out and smoothed out with a curve. It shows increasing stretch in the high treble. Why?

Brian Capleton says, "Musical pitch itself is a subjective perception, not a scientifically measurable, physical quantity." Frequency is a scientifically measurable, physical quantity, but pitch is a function of the human brain, and therefore, it is not consistent, not linear, not the same from individual to individual. We will never agree how much to stretch the high treble. Get over it.

But it is clear that the highest pitches of the piano must be stretched to satisfy our imperfect human sense of pitch.

But there is a problem, because you have the mid-range which may be tuned in fairly linear objective fashion, and you have the high treble which must be tuned rather subjectively.

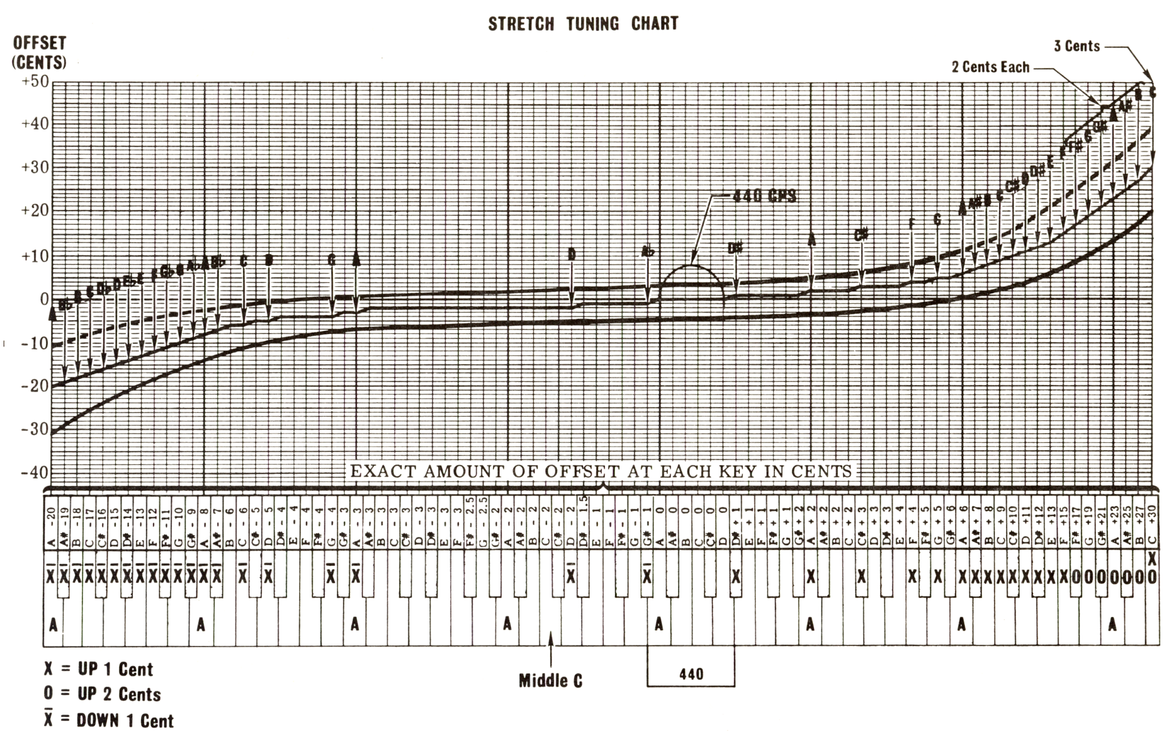
How does one go about connecting the two?

To avoid the tuning sounding flat in the high treble, we might wish to tune, say, pure triple octaves, or for that matter even expanded triple octaves on some pianos, but in order to do so, we must distribute the stretching as we tune up into the treble, sharing the stretch between the 3 individual octaves, rather than trying to make the stretch all in the top octave, which would cause excessive beating in the top octave.

Connecting the linear mid-range with the subjective high treble is a great source of disagreement among piano techs over how to go about this. Since the high treble is subjective, then by necessity the transition to it must also be subjective.

Here is a Rhodes electric piano sound. This first tuning is a straight 12th root of two unstretched tuning. Anyone like this high treble? [Play chord of nature check to the top.]

In the 1970's, Rhodes published a stretch tuning for their electric piano.



I like the tuning so much that I have normalized it and smoothed it out for today's more accurate electronic tuning devices. Much like the harpsichord the Rhodes piano has nominally zero inharmonicity. One of the reasons harpsichords need no stretch is their constricted range. They don't generally play high enough to need high treble melodic stretch. But the Rhodes included 88 note models. And I just described the factory Rhodes tuning as a "stretch" tuning. Why is it stretched? Well, the high treble still needs to be sharp, inharmonicity or no inharmonicity. We do not like to hear the high treble sharp because of inharmonicity; we just want to hear it sharp, because! So to have any hope of connecting the linear mid-range with the subjective high treble, they stretch the mid-range by 2 cents per octave. Two cents per octave. That means its wider than the pure 12th equal temperament by a good bit.

Does it hurt anyone's ears to hear that stretch even though there is no inharmonicity? The octaves are stretched 2 cents. The fifths are contracted .79 cents, so the 5ths will sound nigh unto pure. A fourth will beat just slightly slower than the octave with the common lower note. The double octaves are 4 cents wide.

So, stretched tunings even without inharmonicity. Let's take it to an extreme.
[Play audio example of straight line tuning.]

I want to use this tuning to demonstrate the difficulty of smoothly connecting the linear mid-range and subjective high treble. In this tuning C8 is at +30 cents. So if you wish to move from A4 at zero cents to C8 at 30 cents, why not just draw a straight line on the graph and connect the two smoothly? Well, this is probably not the thing to do. There is music that can be played on this, and it probably does not sound as bad as one might expect. However, this is one temperament where the 3rds, tenths, and 17ths really do get too wide.

Connecting the mid-range to the high treble will be a problem regardless of the chosen stretch level because the mid-range stretch cannot possibly be enough to connect the two in a straight line; fudging will be necessary. But theoretically, the higher the stretch in the mid-range the less fudging will be needed to connect to the high treble. Note that this has implications for OTS levels in RCT; if the purpose of stretching the mid-range is to connect the objective mid-range to the subjective high treble, then it does not follow that higher stretch level in the mid-range requires higher and higher high treble. In my opinion, the high treble should be tuned much the same for all stretch levels. We tune the high treble sharp to satisfy human hearing, not because of inharmonicity or the chosen amount of stretch.

OK. So, what are the enemies of perfect tunings? We know inharmonicity is an enemy, although it is not so much inharmonicity itself that causes a problem as it is the change in inharmonicity through the scale. Don't take my word for it. Here is Brian Capleton:

"Even if inharmonicity precludes very good progression, a good solution must nevertheless be found. Neither a good progression of beat rates in the thirds at the expense of the fifths, nor a good set of fifths at the expense of the thirds beat rate progression, will constitute a good scale... Just as in the 'traditional' model we distribute the Pythagorean comma throughout the scale, so in the real world we also sometimes have to distribute any 'distance' by which the actual physics of the scale deviates from the 'traditional' model. The main cause of this is rate of change of inharmonicity over the scale, rather than inharmonicity itself."

Add to this change in inharmonicity, the requirements that tunings seem to require both linear objective tuning in the mid-range and subjective tuning in the high treble, and you go beyond difficult to theoretically impossible.

But of course, it gets worse:

Composite beat rates, that is, whole sound beat rates made up of simultaneously-sounding multiple pairs of coincidental partials may give a slightly different impression of the beat rates as compared to any one individual pair of coincident partials. Here is how Virgil Smith put it:

..."Beats can be heard two different ways: between single matching partials, and between notes as the ear hears them naturally with all the partials of each note sounding. They both originate with partials, but they are heard in two different ways. For clarity, one will be referred to as ‘partial beats’, and the other as ‘natural beats’ ... Even though each note contains many different partials, the ear hears the note naturally as one pitch and sound. This sound is referred to as the ‘whole sound’.”

And here is how Brian Capleton puts it:

"...In tuning unisons, octaves and multiple octaves, it is necessary to listen to the whole soundscape without concentrating on any specific adjustable partials in the spectrum."

He continues with a quote that is too long and complex for a class like this, but here it comes anyway, because my point is that Brian Capleton deals with the same subject matter as Virgil, but in a way that is more learned and scholarly. Brian Capleton is worth your time to study. Here goes:

"Where falseness is not prevalent, the Weinreich theory suggests that for small mistunings, not only does beat attenuation occur, but the actual beat rate may be slower than expected by 'traditional' theory... Whatever the actual underlying mechanisms, the results show that judicious tuning can produce results 'better' than that predicted by 'traditional' theory, in terms of the manifest beat rate versus the actual mistuning... It is important to remember that there does not have to be coupling effects in order for beat amplitude attenuation to occur... But coupling effects can produce pronounced attenuation in their own right. The effect of the beat rate being smaller than the mistuning, however, does imply actual coupling effects... [T]he final soundscape of the properly tuned unison can also be 'better' than that of the single string, when the aim is to 'reduce beating' or fluctuations. This is not achieved merely by masking false beating in the single string by an array of further beat patterns in the unison. Rather, the false beat fluctuations that would otherwise be inherited, can themselves be eliminated or reduced in the final spectrum, through judicious tuning."

This is worth your further study!

So why do we stretch? To match inharmonicity, to connect the objectively tuned mid-range to the subjectively tuned high treble, and to tune the whole soundscape to its most consonant and pure.

We stretch because it just sounds better that way. That seems clear enough, but the explanation for why that is can be very difficult as I think we have seen.

But here is a suggestion. Think about the boat video, and the graph on how waves combine. Beats are caused by rising and falling amplitude of the combined waves, but if the constituent waves of three or more sounding notes combine in such a way that one wave is always on an "up" when graphed, then amplitude fluctuations may not happen; beats can be attenuated and even masked, and the resulting sound may have a pure effect.

Various people have made various claims about how various widths of equal temperament promote this masking effect. I am not here today to endorse one equal temperament over another. I tune 19 tone to the pure 12th equal temperament every day, and that sounds mighty pure to me when carefully executed, but I don't know that that promotes the purest sound; I will leave that to others. Perhaps if there were more people actively exploring the wealth of gradations of equal temperament, we would soon establish which, if any, of the gradations produces the purest result.

The key to executing a uniform level of stretch across a whole tuning is to choose a width of equal temperament, learn the beat rate relationships of that width of equal temperament and impose those relationships across an entire scale.

Using beat rate relationships to analyze the width of equal temperament

So, how do we analyze the width of our equal temperament?

If in zero inharmonicity tunings, the 4ths and 5ths are tempered the same, and beat the same, then this means one is tuning a pure 4:2 octave. (Because if the 4ths are faster than the 5ths, that means the 4:2 octave is tempered.)

So, a pure 4:2 octave seems to me to be the relationship necessary to replicate the beat rate of the traditional model. But we often temper the 4:2 octave wide. So this is evidence that we are actually tuning some wider equal temperament than that denoted by the 12th root of 2.

I don't intend this as a definitive pronouncement. I am simply pointing out some evidence that makes me think we don't adhere to 12th root of 2 equal temperament, but rather tune an equal temperament based on a somewhat larger number.

If we are in fact tuning a wider equal temperament that that of the 12th root of two, then we must widen all the other intervals correspondingly to be successful at equal temperament. If stretch and temperament are not coordinated then the temperament is not equal, except by accident.

Why should everyone study the various widths of equal temperament? The same restoration of theoretical beat rates needs to happen in the widened equal temperaments as in traditional equal temperament, and for exactly the same reason, that is, artful tuning emulates the theoretical beat speeds. Inharmonicity will mess up those speeds, but surely no one would suggest that it is a waste of time to know the beat rates from which you are deviating!

Of course, for each width of equal temperament there will be a unique set of theoretical beat rates to be emulated.

And, when one is familiar with the various widths of equal temperament, then one becomes aware that some beat rate relationships are simply impossible. This is important!

If a relationship is impossible in zero inharmonicity, such as the impossibility of having pure 5ths and pure octaves at the same time in equal temperament, inharmonicity will not suddenly make pure 5ths and pure octaves compatible within equal temperament.

Knowing the relationships within zero inharmonicity tuning will keep us grounded, so to speak, and can keep us from using incompatible checks, or from trying to tune an impossible relationship.

As I said, pure 5ths *and* pure octaves cannot coexist within any one width of equal temperament.

You can't have pure 12ths and pure 19ths in any equal temperament.

You certainly cannot have pure 4ths and pure 5ths in any equal temperament.

It would be rare for inharmonicity to improve a relationship. For example, inharmonicity will not erase the comma of Pythagoras. The comma may be lessened, but not erased.

Again, if the fourths are tempered the same as the fifths, then the octave is not tempered/stretched; and if the fourths are in general faster than the fifths, then the octaves are stretched, not pure.

If you like pure 12ths and equal temperament, then you cannot have pure octaves; they must be stretched.

(And parenthetically if you like pure 12ths then you cannot tune pure 5th equal temperament, nor can you tune traditional pure octave equal temperament.)

Undisciplined stretch leads to incompatible beat rate checks and mistakes in temperament.

To the extent that we say temperament octave 4ths should beat 1 per second, and temperament octave 5ths should beat 1/2 beat per second, we are using a check that should more properly be associated with pure 12th equal temperament, than pure octave equal temperament. In pure 12th equal temperament, 4ths must be tempered twice the amount of fifths. Now we have an answer to how much faster to make the fourths over the fifths in tuning the temperament!

In pure 12th equal temperament, the double-octave must be tempered twice that of the single octave. If that is so, then the fourth D3-G3 must beat the same as D3-D5 in order to make the

G3-D5 12th pure. The 4th partial of the D3 is at D5; the third partial of G3 is at D5, and the 1st partial of D5 is at D5. That is, the 3rd partial of the G3 is tuned pure with the 1st partial of D5. Now, to tune the D3, its 4th partial must be tuned 2.52 cents flat to the D5 1st partial; that means that it will also be the same 2.52 cents flat to the G3 3rd partial. In other words, in pure 12th equal temperament, the D3-G3 fourth will beat the same as the D3-D5 double octave. How is that for quantifying stretch?

In pure fifth ET consider the 3:2:1 relationship, for example, C3, G3, G4.

If the 3/2 is tuned pure, then the 1/3 and 1/2 will be tempered the same, that is, the G4 will beat the same with C3 as it does with G3, wherever it is placed.

So you could call this the octave, twelfth test of the pure 5th.

In pure 5th equal temperament, in a root position triad, the minor third beats 1.5 times as fast as the major third, a simple 3/2 relationship.

In pure 5th equal temperament, in a root position minor triad, the two thirds minor and major actually beat at the same rate. (This should be a well-known check to piano tuners, although it checks the 6:4 fifth, not the 3:2.)

In pure 5th equal temperament, in a 1st inversion major triad, the minor 3rd and minor 6th beat nearly the same.

And of course, in pure 5th equal temperament, in an open major triad such as C3-G3-E4, the 6th and tenth will beat the same.

No matter which width of equal temperament you choose, you have to be able to execute that width across a scale. How to do that? We need to use the same checks across the keyboard instead of changing checks with the sections, to promote coherence and unity within a tuning. Checks from different widths of equal temperament must not be mixed! The reward for tuning a single width of equal temperament in this manner will be a purity and consonance of sound that is not achievable any other way.

So, what checks can be used all the way across the keyboard? Well, parallel intervals: octaves, 2:1; 12ths, 3:1; double octaves, 4:1, especially the filled in double octave; 17ths, 5:1, and perhaps even 19ths, 6:1.

Inherent in downplaying inharmonicity’s importance is being careful to conservatively designate which intervals are "tuning intervals". I think there is little or no reason to consider any interval larger than the 19th as a tuning interval.

[Demonstrate.]

In a whole sound context one can use the chord of nature across the whole scale. I hope you are familiar with the treble version of this.

[Demonstrate chord of nature in treble.]

But, using the Chord of Nature in the bass?

[Demonstrate bass version.]

(2-3-)4-5-6-7-8 partials of the note to be tuned.

For F1, play F2, C3, F3, A3, C4, D#4, F4 notes at levels of the partials of F1.

It might seem wrong to include A3, the 5th partial, and D#4, the 7th partial, since they are fast-beating intervals. However, you may just find that the whole sound is quite consonant, pure, and free of obvious beating. Tune the F1 so that when added to the rest, no new beating is added to the overall sound.

Mixing equal temperament widths in a single tuning does not promote purity, but rather promotes chaos in a tuning. And I daresay most tuners do this, in the name of dealing with inharmonicity. I see tuning a single width of equal temperament as an alternative to chaos.

I believe artful tuners already make use of the range of widths of equal temperament that is more than just indefinite stretching. My point here is that given that we seem to use these temperaments already, we should study them, be familiar with them, and be able to accurately execute them in order to provide consistent results across a whole scale -- not to mention the fact that I question the notion that the pure fifth would ever successfully serve the tuning of 12th root of 2 equal temperament.

So, is there a relationship between various widths of equal temperament and stretch? Yes, they are in fact the same thing. Use temperament to execute a chosen, uniform, consistent amount of stretch, and coherence and purity will be promoted over chaos. Stretch can be executed and precisely quantized by temperament.

Also, if stretch and temperament are the same thing, then tuning at your chosen stretch level doesn't begin when tuning octaves out from the temperament, it starts as you adjust the beat rates within the temperament of all the 3rds, 4ths, 5ths and 6ths, before you ever get to any octaves. You'll have an easier time extending the temperament up and down if you have already established your equal temperament width, and hence, your stretch level. Your stretch level must be established within the temperament octave!

We have listened to harpsichord and electric piano which are both zero inharmonicity instruments and demonstrated differing widths of equal temperament. Lightning did not strike as

I put a stretch tuning on a harpsichord; lightning did not strike when Rhodes published a stretch tuning in the 70's for their zero inharmonicity electric piano.

Is it true that tuners are urged to use narrow tunings on high inharmonicity pianos such as spinets and other small pianos? Is it true that tuners are also urged to use narrow tunings on low inharmonicity piano such as Yamahas and Bosendorfers? And is it true that Steinway D's, which are relatively high inharmonicity are regularly stretched the most of all for the concert hall? Who makes this stuff up? In what universe does this all make sense? Perhaps inharmonicity is not a determining factor in how much to stretch.

I always disliked tuning narrow for spinets because it was so different than that to which I was accustomed in larger pianos. And now I know it was OK for me to not like those narrow tunings.

Eventually, OnlyPure came along and started tuning every piano to try to make its beat rates conform to the model of pure 12th equal temperament. This means that every piano tuning done with OnlyPure has a characteristic sound. The 12ths are always clean; the fifths and octaves are tempered similarly.

With years of experience under my belt with OnlyPure, I think I can safely state that temperament and inharmonicity have nothing to do with each other. That is, a stretch level as executed by a chosen width of equal temperament can be chosen freely on any piano, regardless of inharmonicity level. If there is an exception to that, I want to know about it! Really. Please.

Macro/Micro, Individual Intervals/Whole Sound

There is a difference between the overall effect of a tuning and the individual intervals that form that tuning. The question is not how to make the best sounding octaves; the question is what width octaves makes the best overall tuning. The question is how can I tune this note to fit in the best with the rest of the piano? The best example I have is a story I have told many, many times, so forgive me if you have heard this. I once took a new ETD to a master tuning session, and did the preliminary tuning with the ETD. The committee came in to begin the master tuning. The CTE in charge played a few intervals in the temperament area and said, "Oh, that isn't the way I like to hear those." I replied that he had not yet checked the overall temperament. When he did check the temperament, he could find no mistakes, even though those intervals he had started with still didn't sound right to him.

In evaluating a tuning, we often listen to an interval by itself, that is, in isolation. Isolated intervals may be necessary while executing a tuning, but they may not be the best way to evaluate a tuning. I believe the test of a tuning should be the overall effect of its whole sound; the whole sound is mostly what we hear when listening to music. Certain widths of tuning seem to lend themselves well to the whole sound, so that when 3 or more notes sound, the beating can be masked and a pureness is heard; 12-tone to the pure octave ET may not lend itself to the pure whole sound effect.

Individual intervals and their beat rates cannot prove a good tuning, only a bad one. The correct proof is the overall effect of the whole sound. 8^)

The point is never individual intervals. You would never say for example that this tuning is superior because of the pure 12ths.

The point is the overall effect of the tuning in musical contexts. The point is the whole sound of notes in combination.

" The weakness in any set of specified beat rates lies in the false idea that good tuning can be ultimately defined by exactly prescribed beat rates, and also in the equally false idea that beat rates in natural piano tone are always constant, and therefore specifiable as a precision rate."

-- Brian Capleton

I once wrote a Journal article about the Accutuner and tried to introduce the concept of macro versus micro tuning. (Microtuning has since come to refer to non-equal temperaments, so the macro versus micro may not be the best way to talk about individual intervals within a whole tuning.) You cannot obsess over individual intervals to the exclusion of how all the individual intervals fit into the overall tuning.

Modern Equal Temperaments and ETD's

OK, I can hear you thinking that widths of equal temperament are old news, that CyberTuner has its OTS series and the Accutuner has its DOB Double Octave Beat setting. Their versions of the various widths of equal temperament are based on brute force expansions of the octave, adding 1 cent per octave or 1 1/4 cents per octave, for example.

And I hear you saying but look at how well ETD's work. Indeed, they do work well. However, their weak link is that they must assume uniform, smoothly progressing inharmonicity. The mistakes that ETD's *do* make tend to be the result of this. The emphasis in ETD's is in correcting for inharmonicity and drawing smooth curves through the scale in order to accomplish that correction for inharmonicity. Its emphasis is not changing between models of equal temperament; I am the one saying that we need to do *that*! With the exception of OnlyPure, which works rather differently from the others, ETD's are not about executing the beat rate relationships of specific widths of equal temperament. They tune smooth curves, and by doing so, they theoretically lay bare the random variations in inharmonicity in the piano being tuned.

Just one more comment about OnlyPure, since I did mention that it works differently from the other electronic tuning devices. OnlyPure does not parse out the individual partials of the notes it is attempting to tune. It operates instead on the whole wave, rather than measuring single partials. So, let's see. That's whole sound from Virgil, whole soundscape from Capleton, and whole wave from Stopper. Hmmm.

RCT Octave Tuning Styles

I went ahead and took a look at the various OTS levels of CyberTuner and tried to relate them to the various widths of equal temperament. [Use RCT graphs of modern ET's.] OK, so OTS 1 is pretty close to 12th root of 2 equal temperament, but it is actually stretched just slightly. And OTS 2 isn't far from the Circular Harmonic System, CHAS. The default standard OTS, OTS 4 does indeed stretch about 1 cent per octave, interestingly, just like Capleton claims is standard.

OTS 7 is fairly close to a pure 12th, 19th root of 3 equal temperament, although it does much better if you turn off Smart Partials, and use the 6th partial in both the low bass and bass, the third partial in both the low tenor and tenor, and then use the 1st partial in the treble and high treble, that is, you'll be using partial 1 from A4 up.

Now, a word about OTS 9. People apparently have the impression that OTS 9 is so wide that it isn't usable, but you know what? It has a stretch of somewhere around 2 cents per octave, the same as the Rhodes factory tuning! Believe me when I say, there is nothing extreme about OTS 9. I actually wish there were wider OTS's, up to an OTS that tuned a pure 5th equal temperament. Such a temperament would have a place in the music world, and would certainly be of historical interest, if nothing else.

Just remember, the wider the equal temperament, the easier it should be to connect the linear mid-range with the subjective high treble. As I showed with that Rhodes tuning, there is certainly a limit to that, but just keep in mind that pure 5th equal temperament expands the octave by 3 1/3 cents, while my straight line equal temperament expanded the octave by about 12 and 1/2 cents. And even then there seemed to be music that could be played on it. <grin>

Ignoring Inharmonicity

I believe that in general we over-emphasize the effect of inharmonicity. As we have said, we should try to restore the beat rates of zero inharmonicity and give the illusion of there being no inharmonicity. It is an illusion, but it is an effective one. We shouldn't worry when beat rates don't match our expectations, because we expect inharmonicity to mess things up. But I wonder if we haven't gone too far. By using the correct model, that is, a single width of equal temperament from bottom to top in each piano scale, we may be able to minimize the differences between our targets and reality.

The over-emphasis on inharmonicity has given rise to an incredibly complex set of "rules of thumb" governing how we deal with inharmonicity while tuning. You all know these rules of thumb, or some or most of them. An example might be how we emphasize the 4:2 in the mid-range and transition to the 6:3 in the bass. As useful as these rules of thumb seem to be, I strongly believe that each piano must be tuned in and of itself without preconceived notions. Rules of thumb are inherently preconceived notions that can, if followed slavishly, lead us into tuning mistakes, especially when we are evaluating the macro tuning, that is, the overall effect of a whole soundscape in the tuning. I hope lightning doesn't strike me, but another example of a rule of thumb is the generalization that fourths don't progress, because on some pianos they may very well do just exactly that.

On the one hand, we try to emulate the beat rates of the math model of our chosen width of equal temperament. Inharmonicity will prevent you from exactly replicating the beat rates of the model, but you keep in mind the beat rates of the model as you tune and make your best fit compromises. Making intervals comply with preconceived ideas of desired width will probably degrade equal temperament.

On the other hand is the method that is more common and, when you think about it, is infinitely more complicated. We have developed extensive rules of thumb based upon our experiences with hundreds and thousands of pianos about how inharmonicity behaves and warps piano scales, and we employ these rules of thumb in our tunings to try to outsmart inharmonicity.

But there is a problem. Just because our experience comes from many, many pianos, when we actually tune we do so only one piano at a time. And inharmonicity varies from piano to piano. Excuse me but broad generalizations about how inharmonicity behaves in pianos may say little or nothing about how inharmonicity behaves in the piano in front of us.

I believe these rules of thumb about inharmonicity may be the largest single source of tuning errors in advanced level tuning.

In my opinion, if we focused on the whole sound of the overall tuning, rather than the individual intervals within that tuning, we might find it easier to agree that one tuning is better than another. As long as certain intervals must sound a certain way, then there will be obvious disagreements among tuners as to what constitutes an ideal tuning. One of the reasons we are so accepting of different tuning preferences is the fact that we focus so much on individual intervals. We should listen to the whole soundscape and evaluate *that*. We should listen to the overall design and effect of the whole tuning, for example, by listening to the sound of an arpeggio with the damper pedal held down.

[Play audio example.]

These examples are two very fine tunings, one OnlyPure and one is another ETD. The differences are not the same as it is on real pianos. So I cannot really demonstrate the OnlyPure effect very well with a computer piano. I think what may be missing here is the effect of soundboard/bridge coupling, and the the partial reinforcement that comes with sympathetic vibrations, that is, the excitement of coincident partials in notes other than the ones actually being played. So I will at some point try to do this example with real pianos. I will have to find a way to create laboratory conditions. Since the non-OnlyPure tuning has fared so well here, I will go ahead and tell you that this is RCT with smart partials at OTS 4.

Little can be determined by listening to, for example, a 3rd-10th test of an octave, because without context you do not know what an individual interval should sound like. You may be able to find an obvious error by listening to a 3rd-10th, but you cannot verify anything as absolutely perfectly *in tune* by listening to a 3rd-10th. The ideal tuning may very well contain individual intervals that may not seem ideal when listened to in isolation. To repeat, a 3rd-10th may be able to detect an error, but it cannot by itself detect perfection.

Here is a bottom line for understanding various widths of equal temperament. How can you evaluate the extent to which inharmonicity is changing the beat rates on you, unless you first are completely familiar with the beat rates that would be present without inharmonicity?

We have allowed inharmonicity considerations to predominate in our tuning. And of course, dealing with the effects of inharmonicity are vitally important.

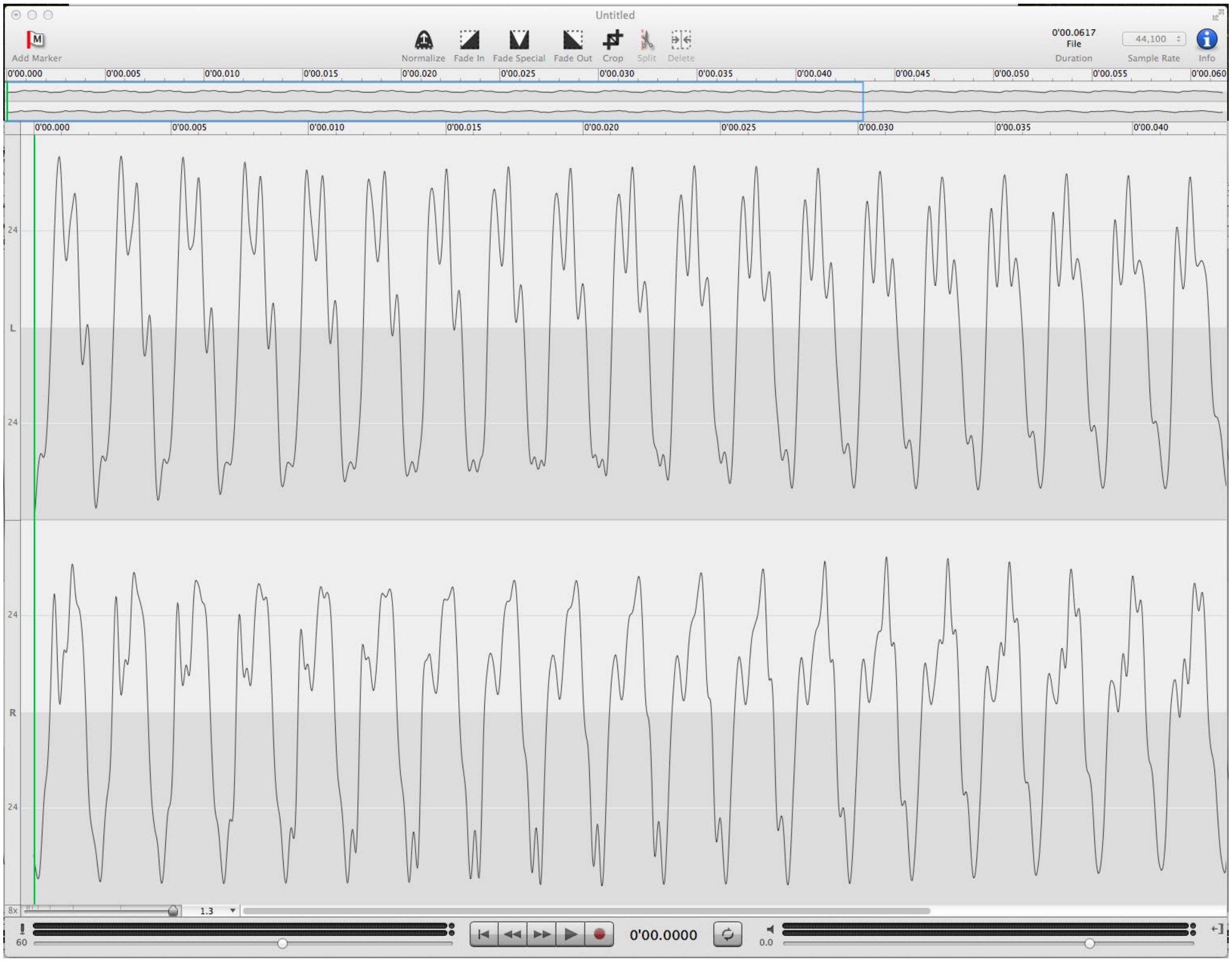
The best description of the effect of inharmonicity is that of a smearing effect. Instead of having intervals with crystal clear relationships, intervals instead have differing relationships all at once. The common example is that of the octave with a wide 2:1 relationship, a just or pure 4:2 relationship, and successively more narrowed relationships as one moves up the partial ladder -- and all these wide, pure, and contracted relationships happen all at once, that is, they are all smeared together. In "smearing" the partial relationships, a random component is introduced — not consistent, but random. The finest tuning must take into consideration the random nature of inharmonicity. So when we hear the whole sound of one interval we hear a smear of relationships, but our human intelligence apparently simplifies what we hear in order to make sense of it all. The question that Virgil Smith brilliantly asked was, can we use that distilled version of that smeared effect to construct a coherent, consonant tuning, that is, concentrating on the composite effect of the whole sound? I am beginning to think that the answer may be yes!

Just because inharmonicity smears our tunings, it does not follow that we must change the underlying model for our tunings. We simply tune a smeared version of our chosen width of equal temperament.

Smeared 12th root of two, smeared pure 5th ET, smeared pure 12th ET; even though they are smeared up when we are done, there are still good reasons to pick one over the other, depending upon the specific situation.

The smearing effect of inharmonicity extends to individual tones.

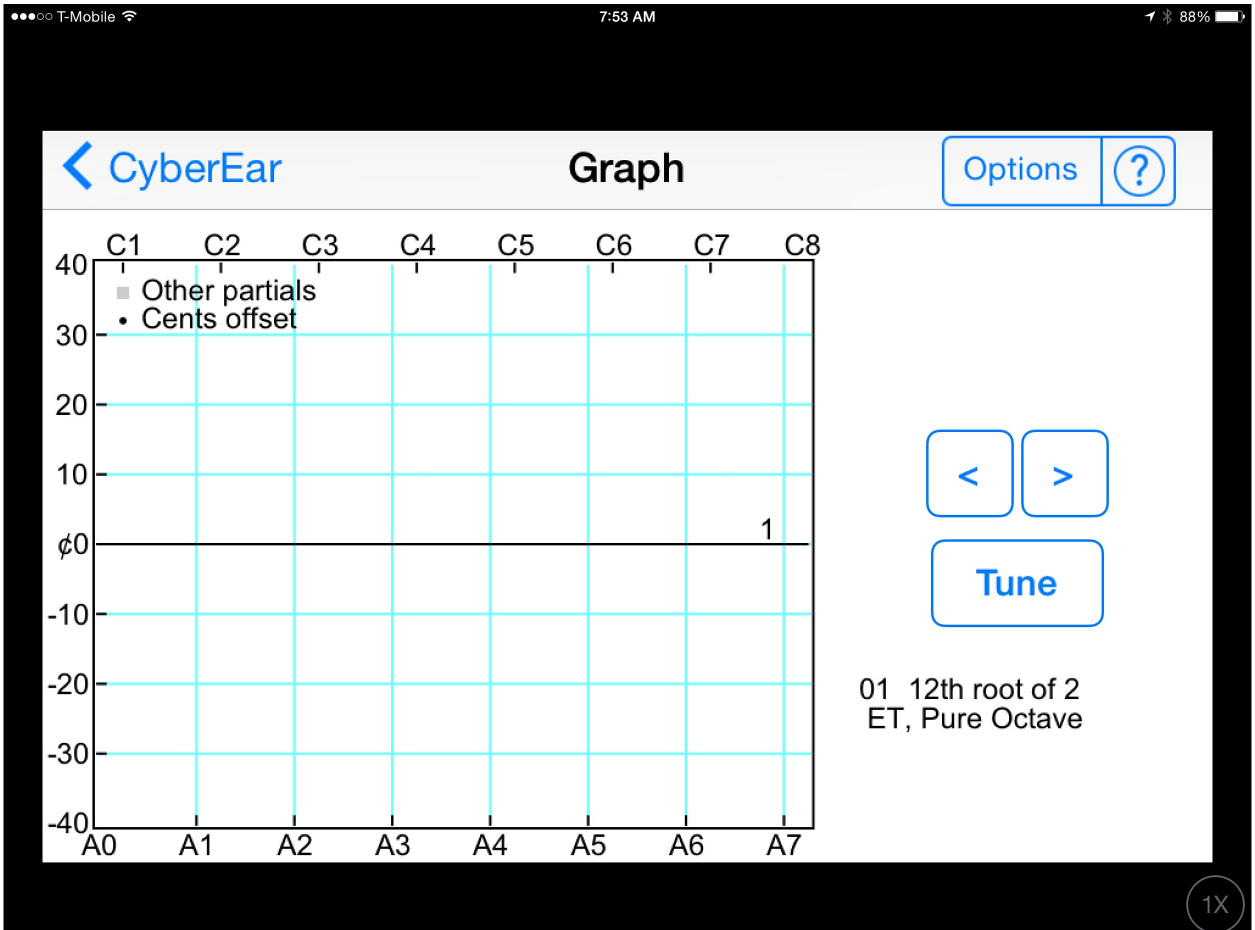
As is shown in graphs the sound of a single piano note being played and held, inharmonicity causes partials to be out of phase with one another and somewhat ambiguous in pitch.



I believe the most pure, most coherent tunings do not necessarily come from being conscious of inharmonicity and for example carefully balancing 4:2 and 6:3 octaves in our tunings, but rather coherent tunings come from chasing the beat rates of your chosen width of equal temperament, as if there were no inharmonicity at all. Just as an experienced tuner may be able to make false beats “disappear”, he and she may be able to make inharmonicity, in effect, disappear as well.

Thanks!

Below, graphs of various widths of equal temperament. Zero inharmonicity assumed: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Tune
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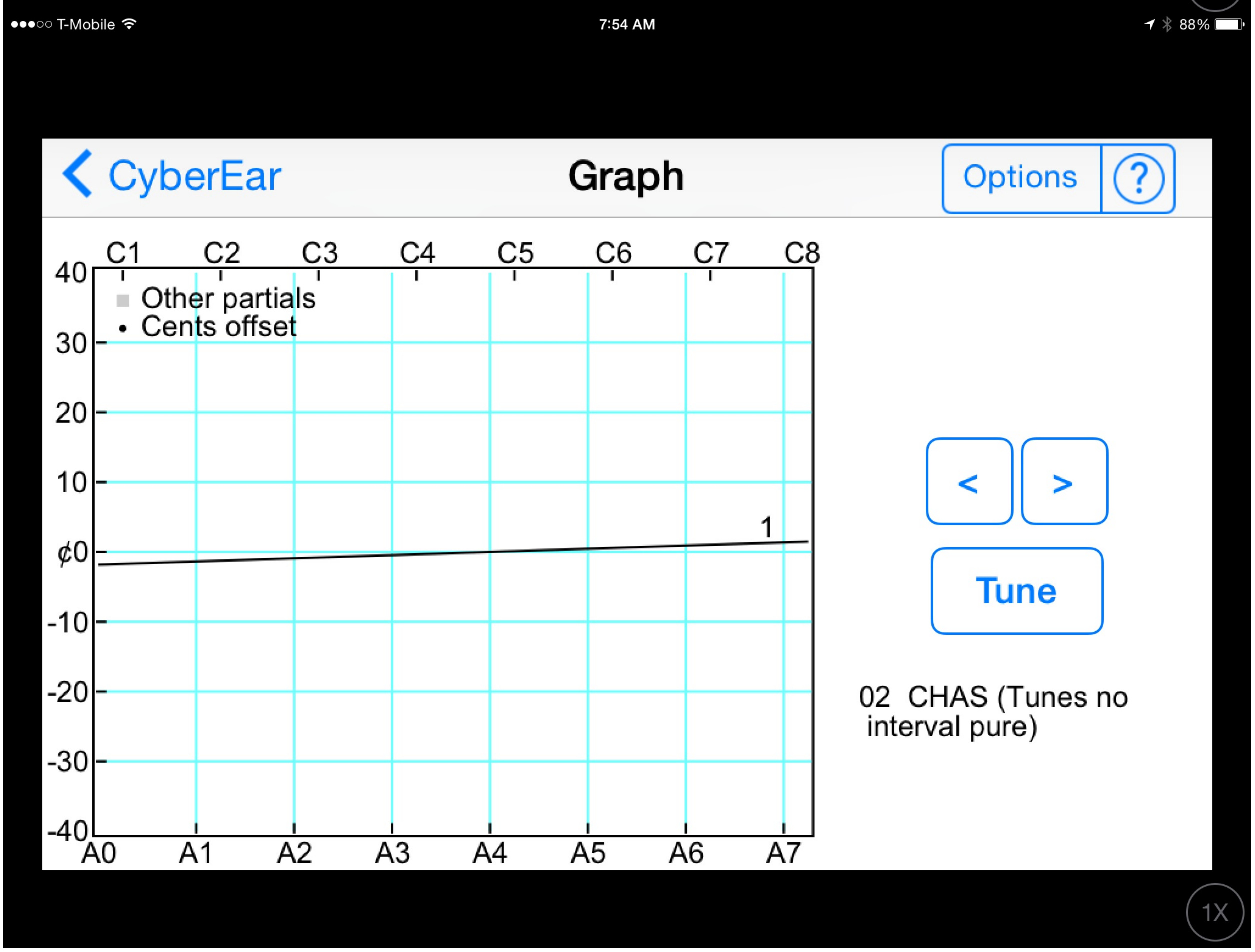
ET, Pure Octave
1X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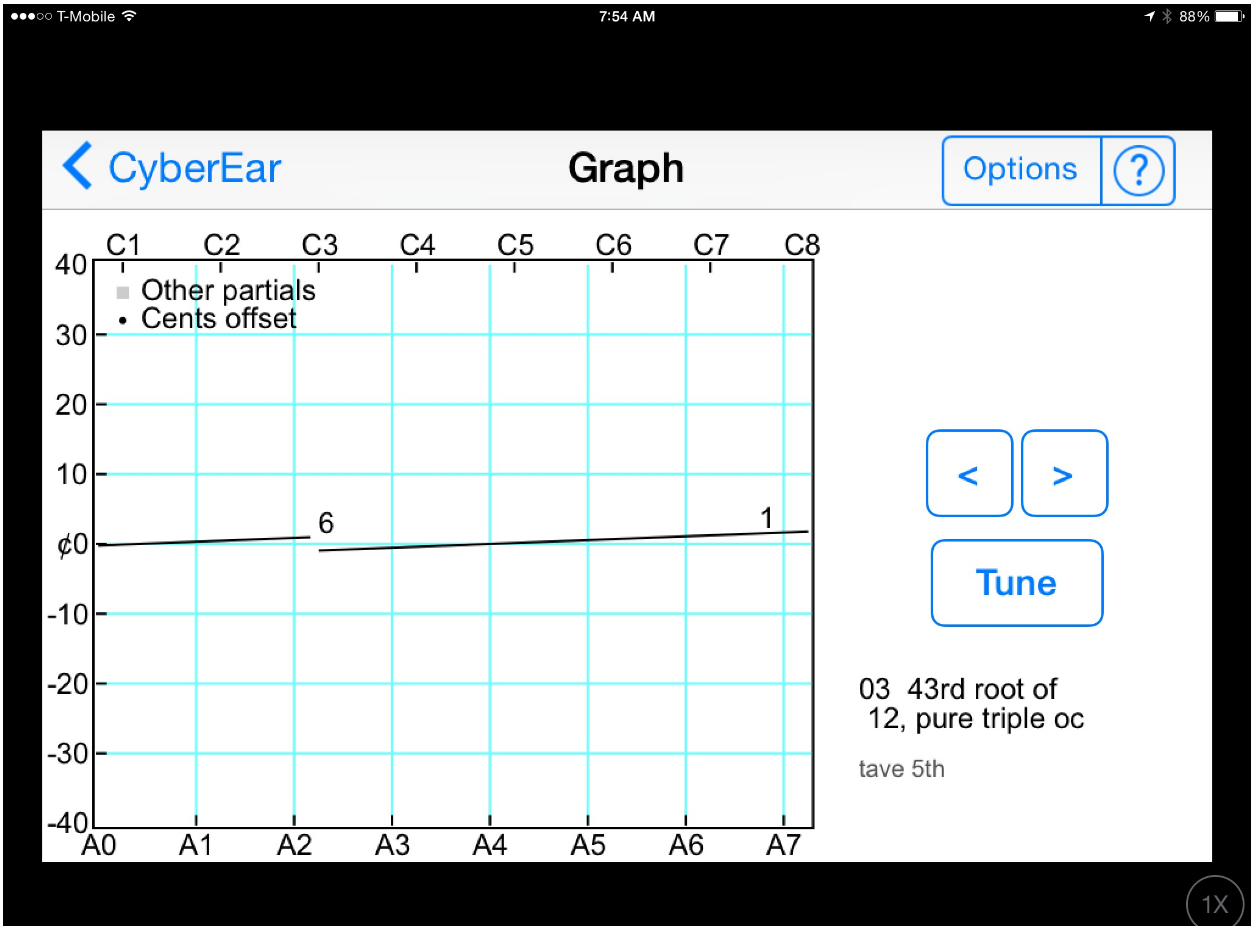
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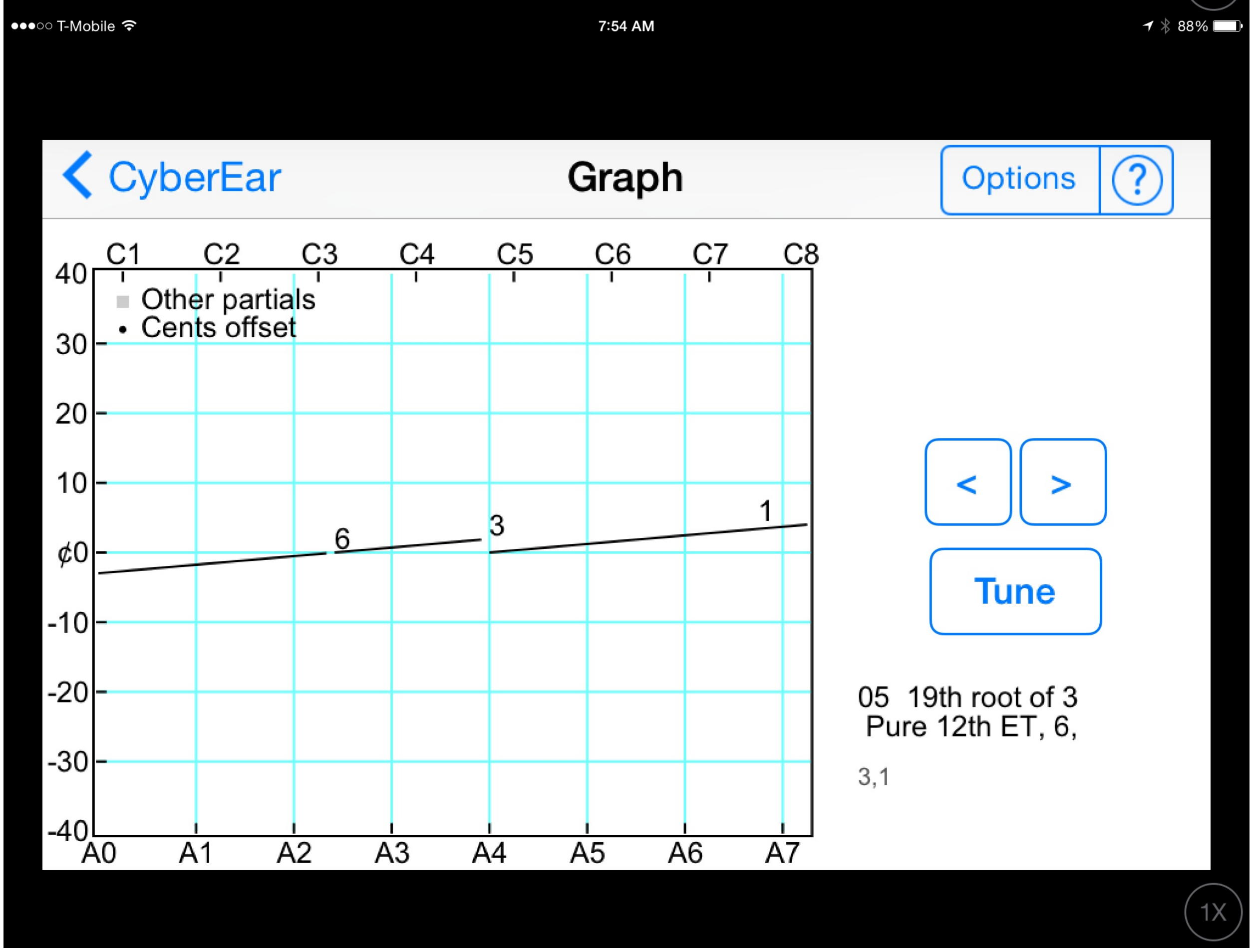
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Tune
05 19th root of 3
Pure 12th ET, 6,
3,1
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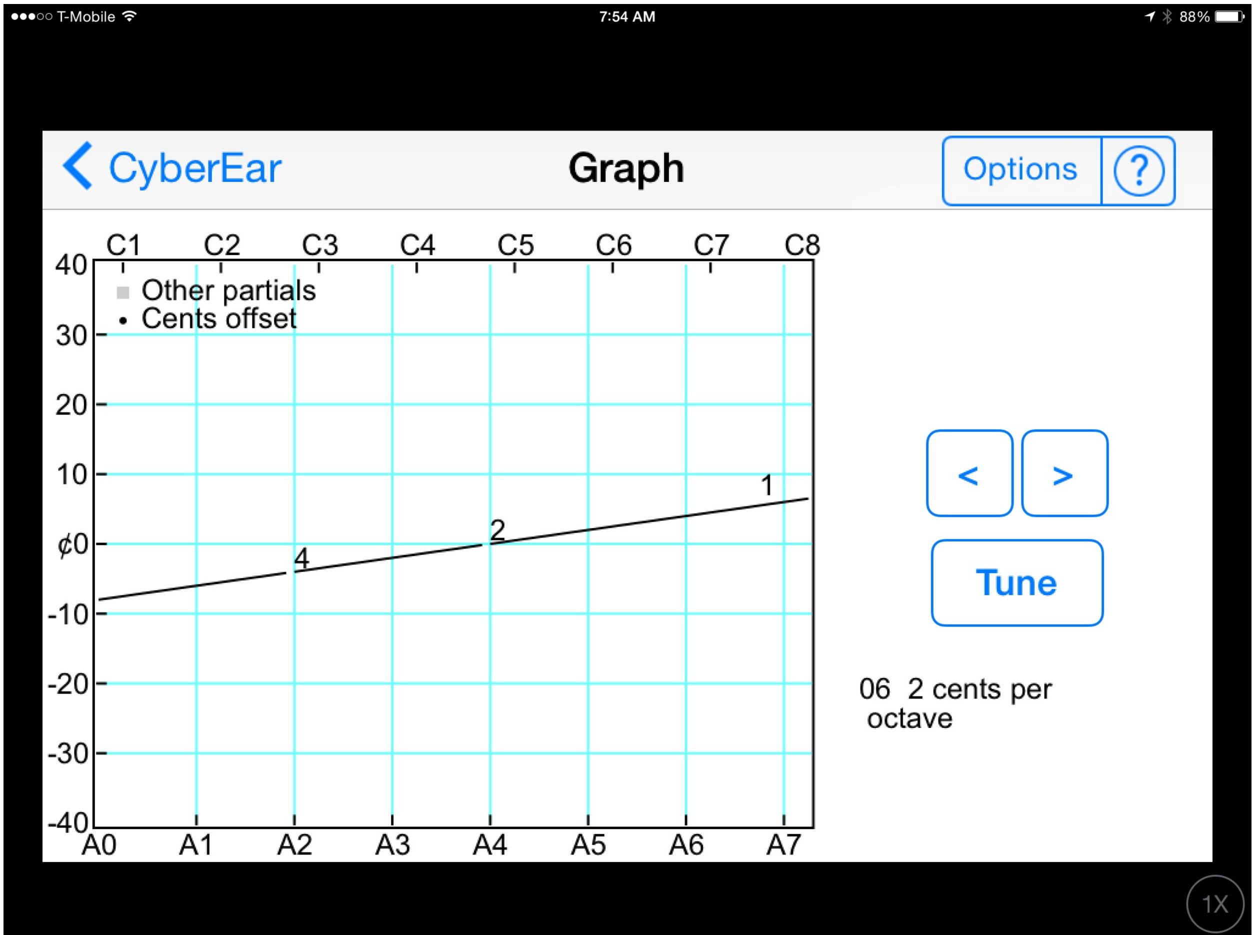
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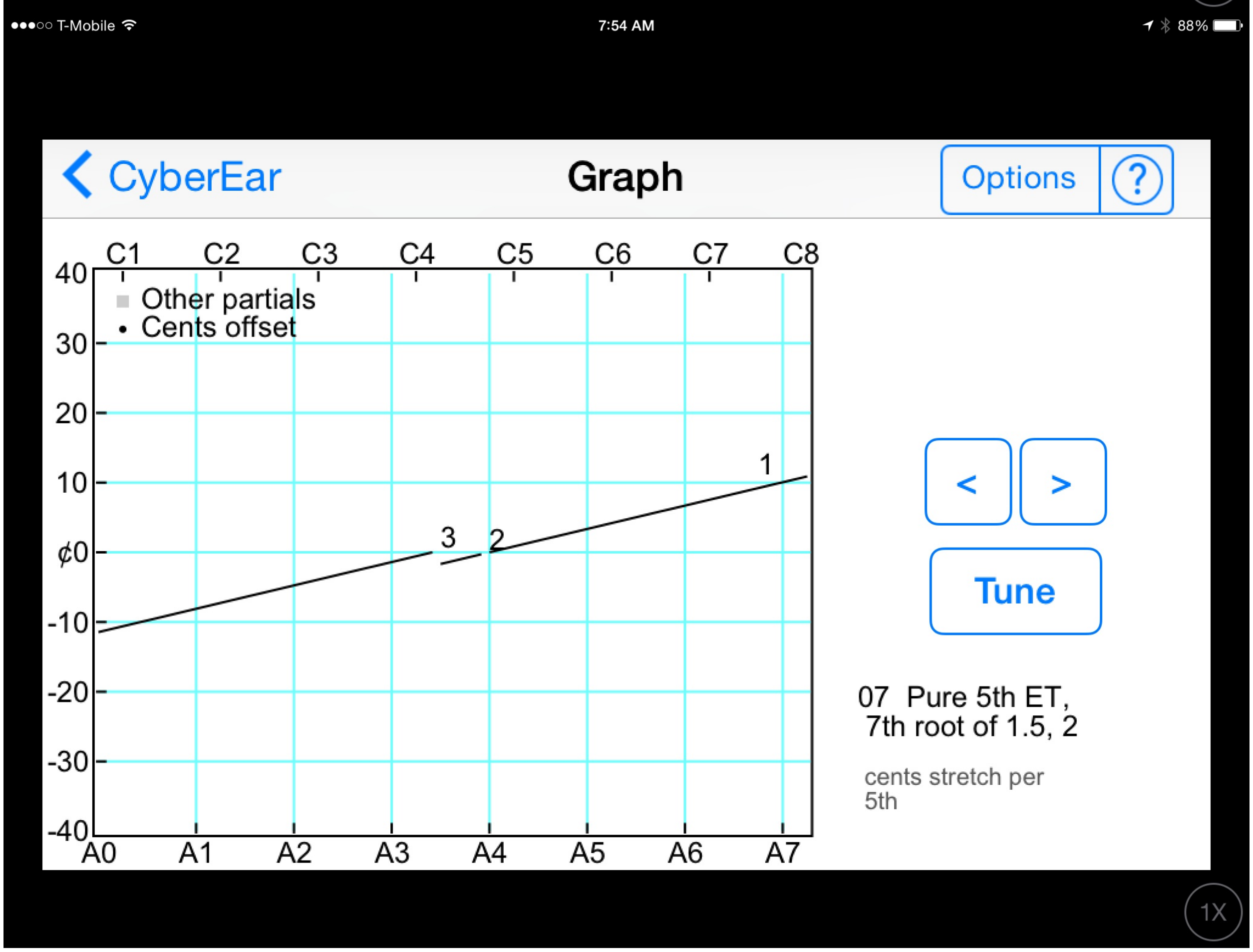
06 2 cents per octave
1X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Tune
07 Pure 5th ET, 7th root of 1.5, 2
cents stretch per 5th
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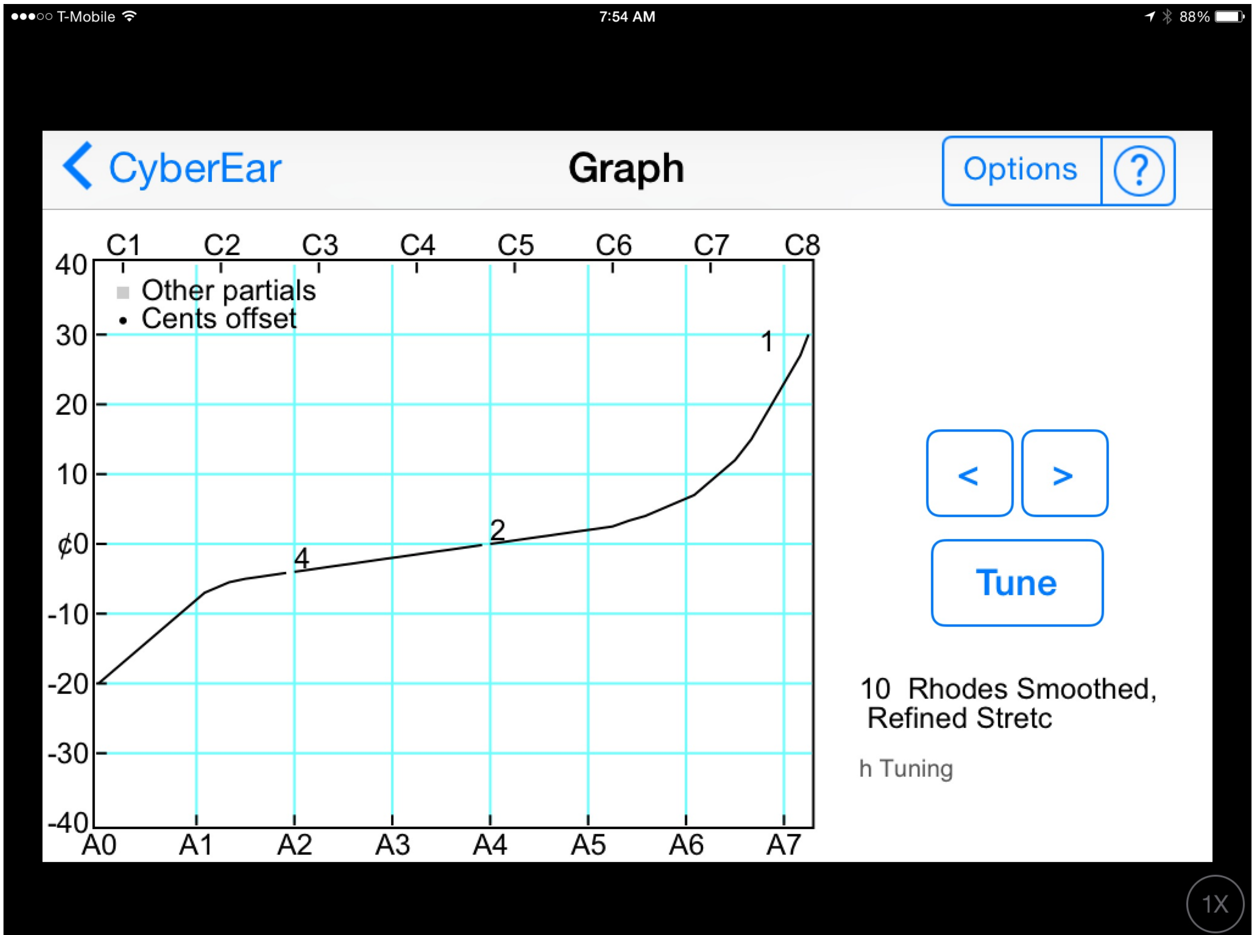
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Tune
10 Rhodes Smoothed, Refined Stretc
h Tuning
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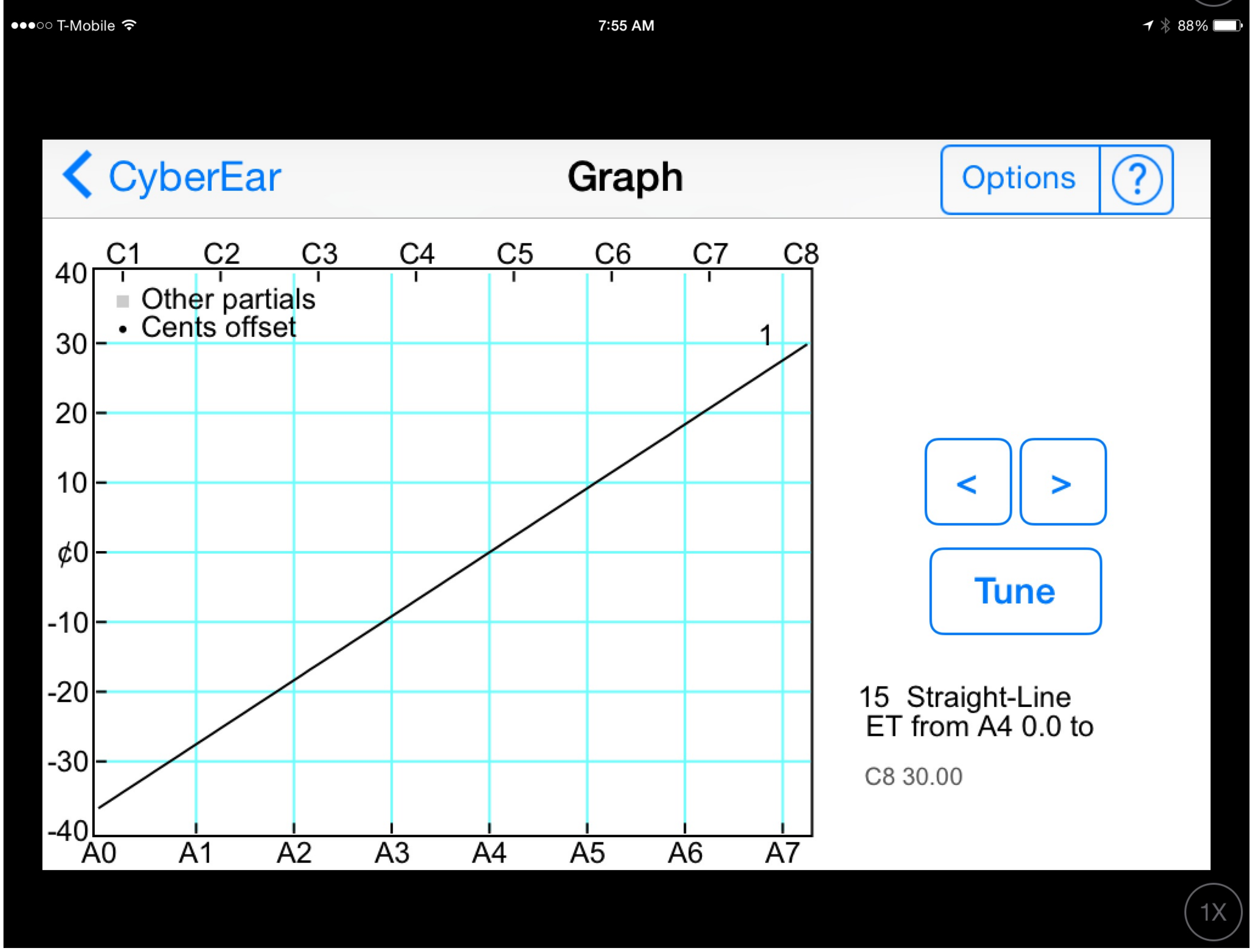
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A0 A1 A2 A3 A4 A5 A6 A7
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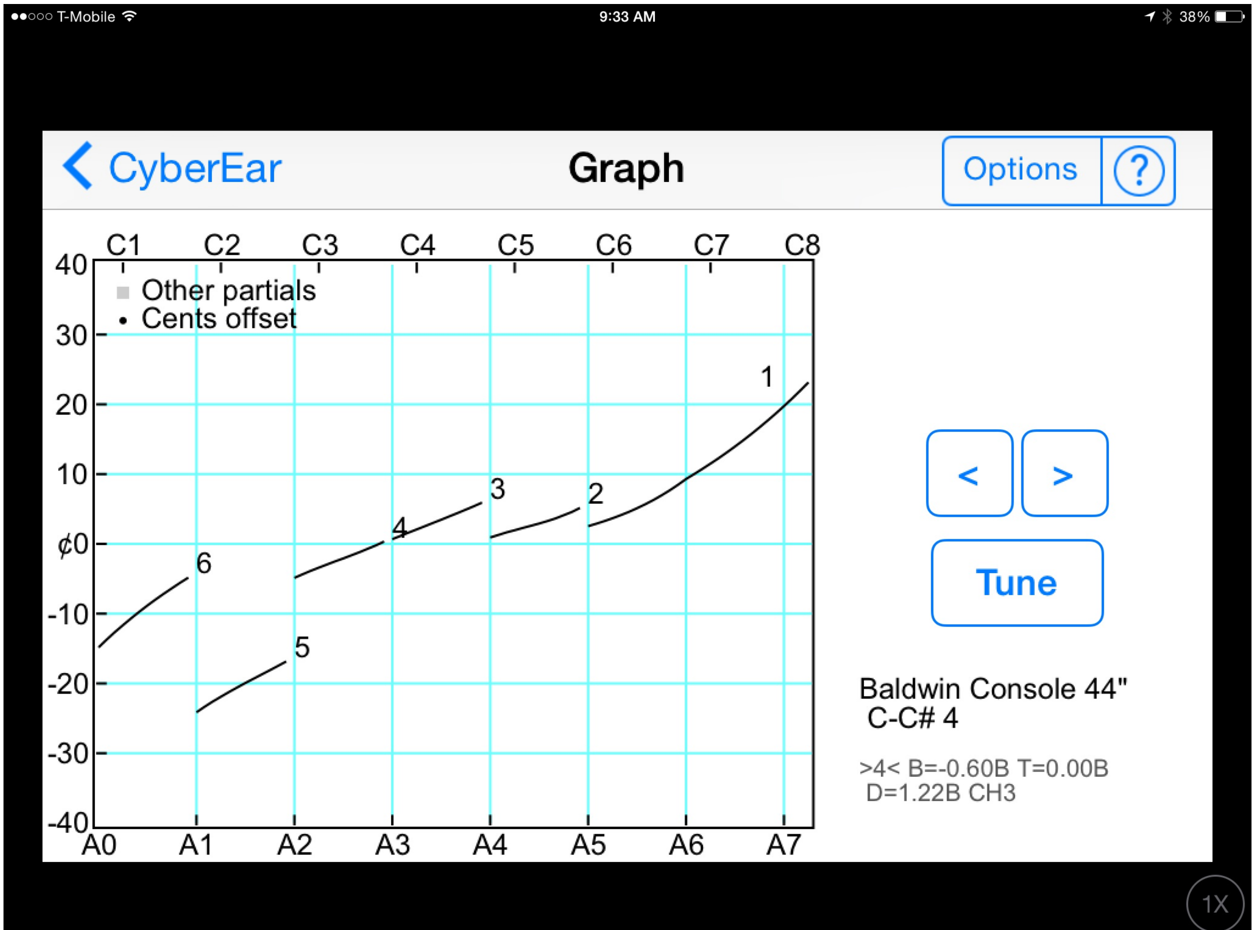
Tune
15 Straight-Line ET from A4 0.0 to
C8 30.00
1X

And finally, on the next page: Higher inharmonicity and wider equal temperament appear similar on a graph; both involve steeper curves. Two graphs, one low inharmonicity, one high:
The graphs have different slope due to differences in the inharmonicity, but both tunings are stretched the same, about one cent per octave. The total steepness of a tuning graph then is the result of the correction for inharmonicity plus the additional stretch of the chosen width of equal temperament.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A0 A1 A2 A3 A4 A5 A6 A7
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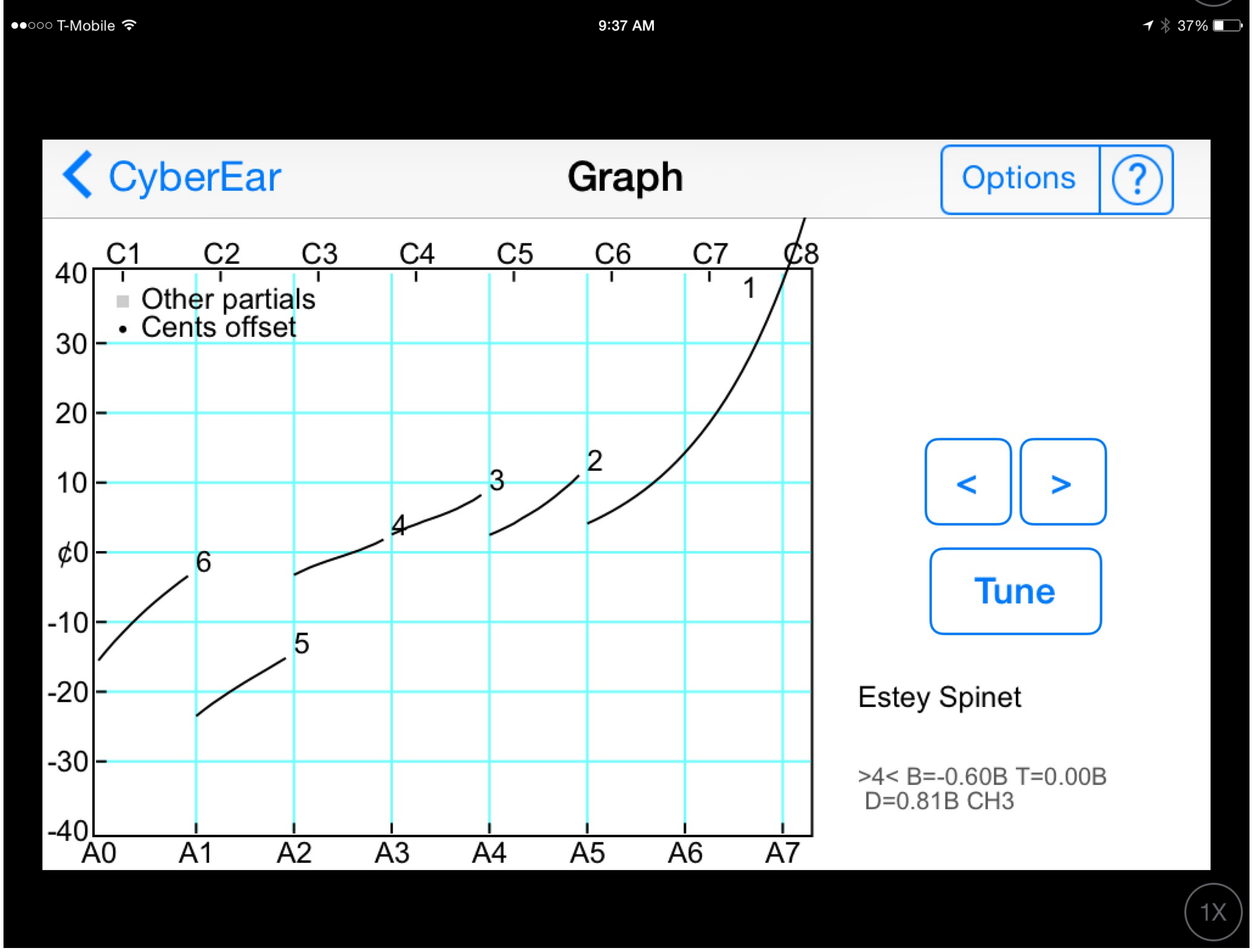
Tune
Baldwin Console 44"
C-C# 4
>4< B=-0.60B T=0.00B
D=1.22B CH3
1X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